Eta-Expansions in Dependent Type Theory — The Calculus of Constructions

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Abstract. Although the use of expansionary η-rewrite has become increasingly common in recent years, one area where η-contractions have until now remained the only possibility is in the more powerful type theories of the λ-cube. This paper rectifies this situation by applying η-expansions to the Calculus of Constructions — we discuss some of the difficulties posed by the presence of dependent types, prove that every term rewrites to a unique long $\beta\eta$-normal form and deduce the decidability of $\beta\eta$-equality, typeability and type inhabitation as corollaries.

1 Introduction

Extensional equality for the simply typed λ-calculus requires η-conversion, whose interpretation as a rewrite rule has traditionally been as a contraction $\lambda x : T.fx \Rightarrow f$ where $x \not\in \mathtt{FV}(t)$. When combined with the usual β-reduction, the resulting rewrite relation is strongly normalising and confluent, and thus reduction to normal form provides a decision procedure for the associated equational theory.

However η-contractions behave badly when combined with other rewrite rules and the key property of confluence is often lost. For example, if the calculus is extended by a unit type 1 with associated rewrite rule $t \Rightarrow *$ (providing t has type 1), then the divergence $\lambda x\!:\!1.\,* \Leftarrow \lambda x\!:\!1.fx \Rightarrow f$ cannot be completed. Another area where η-contractions cannot be used is where the lambda calculus is enriched with base constants and associated rewrite rules which may be confluent on their own, but which fail to be so in the presence of a contractive η-rewrite rule. For example, if we regard 1 as a base type with constants $f : 1 {\rightarrow} 1$ and $* : 1$ and with rewrite rule $fx \Rightarrow *$, then $\Rightarrow$ is confluent. However, as the above counterexample shows shows, the combination of $\Rightarrow$ with the contractive η-rewrite rule fails to be confluent — see [5] for further details.

Recently several authors [1, 4, 6, 14] have accepted the old proposal [13, 16, 17] that η-conversion be interpreted as an expansion $f \Rightarrow \lambda x.fx$ and the resulting rewrite relation has been shown confluent. Infinite reduction sequences such as

$$\begin{array}{llll} \lambda x.t & \Rightarrow \lambda y.(\lambda x.t)y & \Rightarrow \lambda y.t[y/x] \equiv \lambda x.t \\ tu & \Rightarrow (\lambda x.tx)u & \Rightarrow tu \end{array} \tag{1}$$

are avoided by imposing syntactic restrictions to limit the possibilities for expansion; namely λ-abstractions cannot be expanded, nor can terms which are applied. This restricted expansion relation is strongly normalising, confluent and generates the same equational theory as the unrestricted expansionary rewrite relation — thus $\beta\eta$-equality can be decided by reduction to normal form in this restricted fragment. In addition, Huet's *long $\beta\eta$-normal forms* [13, 17] are exactly the normal forms of the restricted expansionary rewrite relation and thus η-expansions provide a natural mathematical theory of this important class of terms. Perhaps most pleasingly of all, these properties are maintained if one adds other type constructors, base types and associated rewrite rules [5, 3]. Finally the use of expansions is supported by categorical models of reduction [11] which has led to new expansionary η-rewrite rules for the coproduct and the tensor and !-type constructors of linear logic [12, 11].

This paper extends the initial results in a different direction by generalising the techniques previously developed to the Calculus of Constructions (λC). The major difference between λC and other theories previously studied is the presence of dependent types, leading to several complications

- As a dependent type system, the types of the λC may have terms embedded within them and so λC cannot be separated into firstly a calculus for deriving the types and secondly a calculus for deriving the terms which inhabit these types. Thus all judgement forms (e.g. type inhabitation, equality etc) must be explicitly parameterised by the context in which the judgement occurs.
- Since β and η-redexes may occur in the types of the calculus, terms no longer inhabit unique types and so decidability of $\beta\eta$-equality at the level of types cannot be assumed in proving the decidability of $\beta\eta$-equality on terms. This also means typability and type inhabitation cannot be assumed decidable and hence the reducts of a term, which partially depend on its type, cannot be assumed enumerable.
- In λ-calculi without dependent types all infinite reduction sequences are derived from expansions appearing in the triangle laws given in equation 1. However, in lambda calculi with dependent types there is another source of infinite reduction sequences, namely the domain of the type of the term to be expanded. That is, if we allow expansions

$$\frac{\Gamma \vdash t : \Pi x : A.B}{\Gamma \vdash t \Rightarrow \lambda x : A.tx} \tag{2}$$

  and define the term $B(x) = (\lambda z : X \to X.X)(x)$, then there is a typing judgement $X : *, x : X \to X \vdash x : \Pi z : B(x).X$ and hence an infinite reduction sequence

$$X : *, x : X \to X \vdash x \Rightarrow \lambda z : B(x).xz \Rightarrow \lambda z : B(\lambda z : B(x).xz).xz \Rightarrow \dots \tag{3}$$

  Another problem with the expansions of equation 2 is that β-normal forms are no longer preserved by η-expansion and thus the usual strategy for the calculation of long $\beta\eta$-normal forms no longer works.

The problems associated with the η-expansions of equation 2 lead us to permit a rewrite $\Gamma \vdash t \Rightarrow \eta^A(t)$ iff, in addition to the usual restrictions, the type A is a normal form. Circularity is avoided by axiomatizing the long $\beta\eta$-normal forms of λC and then permitting an expansion only if the domain of the type of the term expanded can be proven to be a such a long $\beta\eta$-normal form.

There have been several attempts to give a satisfactory account of the Calculus of Constructions and other systems in the λ-cube with $\beta\eta$-equality. One of the major problems of doing so is that if one defines $\beta\eta$-equality on pre-terms with the equation $t = \lambda x{:}A.tx$ ($x \not\in \mathtt{FV}(T)$), then the associated equational theory on pre-terms is too coarse, equating semantically distinct entities, e.g.

$$\lambda x{:}A.x =_\beta \lambda x{:}A.(\lambda y{:}B.y)x =_\eta \lambda y{:}B.y$$

where A and B are arbitrary types. However, as η-expansions are best formulated on well-typed terms, we use a typed version of $\beta\eta$-equality and so avoid the inconsistencies inherent in the untyped version. What's more, typed and untyped $\beta\eta$-equality coincide on well-typed terms *of the same type* [9]. Geuvers' paper also proves that typed $\beta\eta$-equality is decidable by orienting the η-equation as a contraction and showing that the resulting rewrite relation, in conjunction with the usual β-redexes, is strongly normalising and confluent. A related paper [7] (which we draw upon) also uses strong normalisation and confluence of β and η-contractions to prove the existence of $\beta\eta$-long normal forms for λC.

After collecting some defintions in section 2, presenting λC in section 3 and defining our restricted rewrite relation in section 4, we prove:

- The methods of [3] are generalised to prove confluence upto an equivalence relation called *type equivalence*. As a corollary we prove that if two long $\beta\eta$-normal forms are $\beta\eta$-equal, then they are the same term.
- We prove that there are no infinite sequences of η-expansions and, since η-expansions preserve β-normal forms, each term normalizes to a unique long $\beta\eta$-normal form.
- Finally, full confluence is proved from which the computability of the long $\beta\eta$-normal form of a term, decidability of $\beta\eta$-equality, typability etc follow as corollaries.

2 Rewriting and Notation

We shall use the following judgement forms:

$\Gamma \vdash t : T$	In context Γ, the pre-term t has type T.
$\Gamma \vdash t =_{\beta\eta} u$	In context Γ, the pre-term t is $\beta\eta$-equal to the pre-term u.
$\Gamma \vdash \mathcal{D}\mathtt{NF}(t)$	In context Γ, the pre-term t is a $\mathcal{D}$-normal form
$\Gamma \vdash t \Rightarrow_{\mathcal{R}} u$	In context Γ, the pre-term t $\mathcal{R}$-rewrites to the pre-term u.
$\Gamma \vdash t \approx u$	In context Γ, the pre-term t is type equivalent to the pre-term u.

We write $\Gamma \vdash J : T$ to indicate that there is a judgement $\Gamma \vdash J$ and that $\Gamma \vdash t : T$ for all the terms mentioned in J, while we write $\Gamma \vdash J_1 \wedge J_2$ if $\Gamma \vdash J_1$ and $\Gamma \vdash J_2$. When the context of a judgement is either clear or unimportant, legibility is increased by omitting the context, e.g. we often speak only of the reducts of a term and omit all reference to the context. We shall use $\eta^A(t)$ as shorthand for $\lambda x : A.tx$ and assume that $x \not\in \mathtt{FV}(t)$.

If R is a rewrite relation, then $R^=$ is its *reflexive closure*, R^+ is its *transitive closure*, R^* its *reflexive, transitive closure* and $=_R$ is its *reflexive, symmetric, transitive closure.* The relation R^{-1} is its inverse, while $R;S$ denotes the forward relational composition of R with S. We often use $\Rightarrow_R$ as alternate notion for a rewrite relation R. A term t is an *R-normal form* if there is no term t' such that $t \Rightarrow_R t'$, while a relation is *normalising* if every term rewrites to a normal form. A relation is *strongly normalising* if there are no infinite reduction sequences $t_0 \Rightarrow_R t_1 \Rightarrow_R t_2, \ldots$. A relation $\mathcal{R}$ satisfies the *diamond property* iff $\mathcal{R}^{-1}; \mathcal{R} \subseteq \mathcal{R}; \mathcal{R}^{-1}$ while $\mathcal{R}$ is *confluent* iff $\mathcal{R}^*$ satisfies the diamond property.

In the rest of this section we briefly review a nice proof of confluence for expansionary $\beta\eta$-rewriting in the simply typed λ-calculus, discuss the problems of applying this proof to λC and use this as motivation to prove an abstract confluence result which will be used later. When considering rewrite relations composed of two fragments R and S (thought of as β-contraction and restricted η-expansion), then confluence of the combined system is often easily established by proving that R and S are separately confluent and that R and S *commute*

$$Com(R, S) \text{ iff } (R^*)^{-1}; S^* \subseteq R^*; (S^*)^{-1}$$

Unfortunately establishing that R and S commute can be difficult as one must consider not just individual rewrites but also sequences of rewrites. Consequently there is a lot of research into finding sufficient conditions to guarantee that R and S commute and we present one such result from [3, 8]. A relation R is said to *strongly locally commute* over S if and only if

$$SLC(R, S) \text{ iff } R^{-1}; S \subseteq S^*; (R^+)^{-1}$$

Lemma 2.1 *If $SLC(R, S)$ and R is strongly normalising, then R and S is commute.*

Applying lemma 2.1 to the simply typed λ-calculus, one may take R to be β-reduction and S to be restricted η-expansion. That β-reduction is strongly normalising and confluent is well known, while η-expansion is confluent as it satisfies the diamond property. Thus one is left to check that β-reduction strongly locally commutes over η-expansion and, although not trivial because of the restrictions on η-expansion, this is no more difficult than checking local confluence. Unfortunately, the following diagram shows that the strong local commutation property no longer holds in λC.

$$\begin{tikzcd}
(\lambda x : B \to B.x)(\lambda y : B.y) \arrow[rr, "\beta"] \arrow[d, "\eta"'] & & \lambda y : B.y \\
(\lambda x : B \to B.\eta^{B_0}(x))(\lambda y : B.y) \arrow[r, "\beta"'] & \eta^{B_0}(\lambda y : B.y) \arrow[r, "\beta"'] & \lambda y : B_0.y
\end{tikzcd} \tag{4}$$

All we know is that B_0 and B are $\beta\eta$-equal and hence we cannot obtain completions of the right form. Of course, if these terms were simply typed λ-terms then the above diagram would actually be complete as type uniqueness would force $B_0 = B$.

The solution proposed in this paper is to worry about the types later! Thus we define two terms to be *type equivalent* if they are equal apart from their λ-abstractions which may be over types which are only $\beta\eta$-equal. This defines an equivalence relation and confluence is first proved only modulo type equivalence. There is a considerable body of literature [2, 15] concerning rewriting modulo equational theories and we now present those definitions and results required to generalise lemma 2.1 to rewriting modulo equivalence relations.

In the following R and S are rewrite relations while E is an equivalence relation. Define the relation $R/E = E; R; E$. The relation R is *coherent* with E iff $E; R \subseteq R^*; E$ while R is *strongly coherent* with E iff $E; R \subseteq R; E$ (note the similarity between the notions of *coherence* and *bisimulation* from concurrency theory). A relation R satisfies the diamond property modulo E iff $R^{-1}; R \subseteq R; E; R^{-1}$, while R is confluent modulo E iff R^* satisfies the diamond property modulo E. The following lemmas may be proved by simple diagram chases.

Lemma 2.2 *If R is coherent with E, then R^* is coherent with E and $(R/E)^* \subseteq R^*; E$. If R is strongly normalising and R is strongly coherent with E, then R/E is also strongly normalising.*

Lemma 2.3 *If R satisfies the diamond property modulo E and is coherent with E, then R^* satisfies the diamond property modulo E.*

Lemma 2.4 *Let R be coherent with E. Then R/E is confluent iff R is confluent modulo E.*

Lemma 2.5 *If R is strongly coherent with E, S is coherent with E and $S^{-1}; R \subseteq R^+; E; (S^*)^{-1}$, then R/E strongly locally commutes with S/E.*

3 The Calculus of Constructions — λC

We give a quick presentation of λC [9]. The *sorts* of λC are $\mathcal{S} = \{*, \Box\}$, while $\mathtt{Var}(*)$ and $\mathtt{Var}(\Box)$ are disjoint sets of variables. If x is a variable and $x \in \mathtt{Var}(s)$, this is sometimes indicated by writing x^s. The pre-terms of λC are denoted Λ and are defined by the syntax $T := x \mid s \mid \lambda x : T.T \mid \Pi x : T.T \mid TT$, where x is any variable and s is a sort. The usual notions of *free variable, α-equivalence* and *substitution* are taken for λC and a pre-term is called *neutral* iff it is not a λ-abstraction.

A *pre-context* is a list of pairs of variables and pre-terms, written $x_1 : T_1, \ldots, x_n : T_n$ such that the variables are pairwise distinct. The empty pre-context is denoted [] and the variables declared in a pre-context Γ are denoted $\mathtt{dom}(\Gamma)$. The *typing judgements* of λC are of the form $\Gamma \vdash t : T$ and the *equality judgements* of λC are of the form $\Gamma \vdash t =_{\beta\eta} u$, where Γ is a pre-context and t, u and T are pre-terms. These judgements are generated by the inference rules of Table 4 presented at the end of this paper. A pre-term t is called a term iff there is a typing judgement $\Gamma \vdash t : T$ and, unless otherwise stated, all future concepts are restricted to well-typed terms. The basic meta-theory of λC may be established using η-expansions but due to lack of space we quote results from [9] which are proven using η-contractions. The first lemma concerns substitution.

Lemma 3.1 *Let $\Gamma \vdash u : A$ and $\Gamma \vdash u =_{\beta\eta} u' : A$*

- *If there is a typing judgement $\Gamma, x : A, \Delta \vdash t : B$, then there is also one $\Gamma, \Delta[u/x] \vdash t[u/x] : B[u/x]$.*
- *If there is an equality judgement $\Gamma, x : A, \Delta \vdash t =_{\beta\eta} t'$, then there is also one $\Gamma, \Delta[u/x] \vdash t[u/x] =_{\beta\eta} t'[u/x]$*
- *If there is a typing judgement $\Gamma, x : A, \Delta \vdash t : B$, then there is also one $\Gamma, \Delta[u/x] \vdash t[u/x] =_{\beta\eta} t[u'/x]$.*

One of the most important lemmas concerning PTS's such as λC is the *Generation lemma.*

Lemma 3.2 *Let $\Gamma \vdash t{:}T$ be a typing judgement. Then*

- *If t is $*$, then $T = \Box$.*
- *If t is a variable x, then there is a declaration $x{:}T' \in \Gamma$ and $\Gamma \vdash T =_{\beta\eta} T'$.*
- *If t is $\lambda x : T'.u$, then there are typing judgements $\Gamma, x : T' \vdash u{:}A$, $\Gamma \vdash \Pi x : T'.A : s$ and $\Gamma \vdash T =_{\beta\eta} \Pi x : T'.A$.*
- *If t is $\Pi x : T'.A$, then there are typing judgements $\Gamma \vdash T' : s_1$, $\Gamma, x : T' \vdash A : s_2$, and $\Gamma \vdash T =_{\beta\eta} s_2$.*
- *If t is FU, then there are typing judgements $\Gamma \vdash F : \Pi x : T'.A$, $\Gamma \vdash U : T'$ and $\Gamma \vdash T =_{\beta\eta} A[U/x]$.*

Lemma 3.3 *If there are typing judgements $\Gamma \vdash t : T$ and $\Gamma \vdash t : T'$, then there is also an equality judgement $\Gamma \vdash T =_{\beta\eta} T'$.*

Another property required of λC is known as *Uniqueness of Products*

Lemma 3.4 *If there is a judgement $\Gamma \vdash \Pi x : A.B =_{\beta\eta} \Pi x : A'.B' : T$, then there are judgements $\Gamma \vdash A =_{\beta\eta} A'$ and $\Gamma, x : A \vdash B =_{\beta\eta} B'$.*

In order to prove some further results about the judgements of λC, we use a notion of *context-equivalence.* Two well formed contexts Γ and Δ are said to be $\beta\eta$-equal, written $\Gamma =_{\beta\eta} \Delta$, iff they can be proved to be so by the following pair of inference rules:

$$\frac{}{[] =_{\beta\eta} []} \qquad \frac{\Gamma_0 =_{\beta\eta} \Gamma_1 \quad \Gamma_0 \vdash A =_{\beta\eta} A' \quad \Gamma_1 \vdash A =_{\beta\eta} A'}{\Gamma_0, x : A =_{\beta\eta} \Gamma_1, x : A'}$$

Lemma 3.5 *Context equivalence is an equivalence relation on well formed contexts. If $\Gamma =_{\beta\eta} \Delta$ and there is a judgement $\Gamma \vdash J$, then there is also one $\Delta \vdash J$.*

Equality judgements are sound for typing, i.e. if one term has a type and is $\beta\eta$-equal to another term, then the second term also has that type.

Lemma 3.6 *If there is a judgement $\Gamma \vdash t_1 =_{\beta\eta} t_2$ such that for $i = 1$ or 2, $\Gamma \vdash t_i : T$ then $\Gamma \vdash t_1 : T \wedge t_2 : T$*

The last lemma we require is a technical one which is needed to provide an induction rank for the construction of long $\beta\eta$-normal forms

Lemma 3.7 *There is a well-founded order $<$ on the terms of λC such that*

- *If u is a subterm of t, then $u < t$*
- *If t is a β-normal form, then there is also a β-normal form T such that there is a judgement $\Gamma \vdash t : T$ and $T < t$.*

Proof The proof follows [7] in using an alternate syntax where a type for a term occurs as a subterm of the term. $\square$

4 A Rewrite Relation

Recall that if t and A are pre-terms, then $\eta^A(t)$ is notational shorthand for the pre-term $\lambda x{:}A.tx$ where $x \not\in \mathtt{FV}(t)$ is a correctly sorted variable. The theory of η-expansions as developed in [] shows that an expansion $\Gamma \vdash t \Rightarrow_{\mathcal{F}} \eta^A(t)$ can only be permitted if t is neither a λ-abstraction nor applied to another term. However in dependent type theories these restrictions on the applicability of expansions are insufficient to prevent unwanted reduction sequences such as equation 3. We therefore impose the additional restriction that the term A is a long $\beta\eta$-normal form by using structural and typing criteria to axiomatise the long $\beta\eta$-normal forms of λC. In Table 1, two judgements $\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(t)$ and $\Gamma \vdash \eta\mathcal{F}\mathtt{NF}(t)$ are defined which characterize those terms which are (internal and full) normal forms of the expansionary η-rewrite rules. These two judgement forms are used to define two others $\Gamma \vdash \mathcal{I}\mathtt{NF}(t)$ and $\Gamma \vdash \mathcal{F}\mathtt{NF}(t)$ which characterise the *internal/full long $\beta\eta$-normal forms* of λC. In Table 1, a pair Γ, t is said to be *not expandable* if either t is a λ-abstraction, or there are no judgements $\Gamma \vdash t : \Pi x : A.B$. Note that as type inhabitation is not assumed decidable, the various normal form judgements of Table 1 cannot be assumed decidable.

The long $\beta\eta$-normal forms of λC as axiomatized in Table 1 are used to define a rewrite relation in two stages. Firstly the purely expansionary η-rewrite relation, denoted $\Rightarrow_{\eta\mathcal{F}}$, is presented in Table 2. As in previous works [11, 10], the condition that terms applied to other terms may not be expanded is enforced by simultaneously defining a further subrelation $\Rightarrow_{\eta\mathcal{I}}$ of $\Rightarrow_{\eta\mathcal{F}}$ which is guaranteed not to include top-level expansions. Thus a subterm which is applied to another subterm may be safely $\Rightarrow_{\eta\mathcal{I}}$-rewritten without the risk of generating the second of reduction loops in equation 1. Note that because we cannot assume the long $\beta\eta$-normal form predicate is decidable, the $\Rightarrow_{\eta\mathcal{F}}$-reducts of a term cannot be assumed enumerable.

Table 1. The Predicates $\mathcal{I}\mathtt{NF}$ and $\mathcal{F}\mathtt{NF}$

$$\frac{t \in \mathcal{S} \cup \mathtt{Var}}{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(t)} \qquad \frac{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(t) \quad \Gamma \vdash \eta\mathcal{F}\mathtt{NF}(u)}{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(tu)}$$

$$\frac{\Gamma \vdash \eta\mathcal{F}\mathtt{NF}(A) \quad \Gamma, x : A \vdash \eta\mathcal{F}\mathtt{NF}(t)}{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(\lambda x : A.t)} \qquad \frac{\Gamma \vdash \eta\mathcal{F}\mathtt{NF}(A) \quad \Gamma, x : A \vdash \eta\mathcal{F}\mathtt{NF}(B)}{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(\Pi x : A.B)}$$

$$\frac{\Gamma \vdash \eta\mathcal{I}\mathtt{NF}(t) \quad \Gamma, t \text{ is not expandable}}{\Gamma \vdash \eta\mathcal{F}\mathtt{NF}(t)} \qquad \frac{\Gamma \vdash \eta\mathcal{R}\mathtt{NF}(t) \quad t \text{ is a } \beta\text{-nf} \quad \mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}}{\Gamma \vdash \mathcal{R}\mathtt{NF}(t)}$$

Table 2. Restricted η-expansions

$$\frac{t \text{ neutral} \quad \Gamma \vdash t : \Pi x : A.B \quad \Gamma \vdash \mathcal{F}\mathtt{NF}(\Pi x : A.B)}{\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \lambda x : A.tx} \qquad \frac{\Gamma \vdash t \Rightarrow_{\eta\mathcal{I}} t'}{\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} t'}$$

$$\frac{\Gamma, x : T \vdash t \Rightarrow_{\eta\mathcal{F}} t'}{\Gamma \vdash \lambda x : T.t \Rightarrow_{\eta\mathcal{I}} \lambda x : T.t'} \qquad \frac{\Gamma \vdash T \Rightarrow_{\eta\mathcal{F}} T'}{\Gamma \vdash \lambda x : T.t \Rightarrow_{\eta\mathcal{I}} \lambda x : T'.t}$$

$$\frac{\Gamma \vdash F \Rightarrow_{\eta\mathcal{I}} F'}{\Gamma \vdash FU \Rightarrow_{\eta\mathcal{I}} F'U} \qquad \frac{\Gamma \vdash U \Rightarrow_{\eta\mathcal{F}} U'}{\Gamma \vdash FU \Rightarrow_{\eta\mathcal{I}} FU'}$$

$$\frac{\Gamma, x{:}A \vdash B \Rightarrow_{\eta\mathcal{F}} B'}{\Gamma \vdash \Pi x{:}A.B \Rightarrow_{\eta\mathcal{I}} \Pi x{:}A.B'} \qquad \frac{\Gamma \vdash A \Rightarrow_{\eta\mathcal{F}} A'}{\Gamma \vdash \Pi x{:}A.B \Rightarrow_{\eta\mathcal{I}} \Pi x{:}A'.B}$$

Let $\Rightarrow_\beta$ be the standard β-reduction defined on the set of pre-terms Λ. To keep a standard form of judgement, we introduce a context-parameterised form of β-reduction on typed terms as follows:

$$\frac{t \Rightarrow_\beta t' \quad \Gamma \vdash t : C}{\Gamma \vdash t \Rightarrow_\beta t'} \tag{5}$$

and then define

$$\Gamma \vdash t \Rightarrow_{\mathcal{I}} t' \text{ iff } \Gamma \vdash t \Rightarrow_\beta t' \text{ or } \Gamma \vdash t \Rightarrow_{\eta\mathcal{I}} t' \tag{6}$$

and similarly

$$\Gamma \vdash t \Rightarrow_{\mathcal{F}} t' \text{ iff } \Gamma \vdash t \Rightarrow_\beta t' \text{ or } \Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} t' \tag{7}$$

Lemma 4.1 *Let $\Gamma \vdash t \Rightarrow_{\mathcal{R}} t'$ where $\mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}$. Then*

- *There is a judgement $\Gamma \vdash t =_{\beta\eta} t'$.*
- *If $\Gamma =_{\beta\eta} \Gamma'$, then $\Gamma' \vdash t \Rightarrow_{\mathcal{R}} t'$*
- *If t is a β-normal form and $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} t'$, then t' is a β-normal form*

Lemma 4.2 *If $\Gamma \vdash \mathcal{R}\mathtt{NF}(t)$ for $\mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}$, then there is no term t' such that $\Gamma \vdash t \Rightarrow_{\mathcal{R}} t'$.*
Proof Induction on the proof that $\Gamma \vdash \mathcal{R}\mathtt{NF}(t)$. □

Note however that we cannot yet prove the reverse implication of lemma 4.2 as this would require proving that if there is a judgement $\Gamma \vdash t : \Pi x : A.B$, then there is another judgement $\Gamma \vdash t : \Pi x : A'.B' \wedge \mathcal{F}\mathtt{NF}(\Pi x : A'.B')$. This in turn would require proving that every term reduces to a long $\beta\eta$-normal form and this must be postponed until later.

5 Confluence Modulo Type Equivalence

We shall now prove that $\Rightarrow_{\mathcal{F}}$ is confluent modulo a relation called *type equivalence.* As described in the introduction, our proof of confluence is a generalisation of the equivalent theorem for the simply typed λ-calculus as proved in [3]. The main difference is that confluence is proved only upto an equivalence relation stating that terms may differ upto $\beta\eta$-equality in the types abstracted over. In the next section this partial confluence result is used to show successively that $\Rightarrow_{\eta\mathcal{F}}$ is strongly normalising, that $\Rightarrow_{\mathcal{F}}$ is normalising and hence that the restricted expansionary rewrite relation defined in Table 2 is confluent.

Two terms t and t' are said to be *type equivalent* in a context Γ, written $\Gamma \vdash t \approx t'$ iff they can be proved so by the inference rules of Table 3. The inference rule for Π-types requires the types abstracted over to be only $\beta\eta$-equal rather than type equivalent so as to enable the proof of lemma 5.9 to go through. There is no need for some form of weakening for type equivalence as such a rule is admissable.

Table 3. Type-Equivalent Terms

$$\frac{x \in \mathtt{dom}(\Gamma)}{\Gamma \vdash x \approx x} \qquad \frac{}{\Gamma \vdash * \approx *}$$

$$\frac{\Gamma \vdash A =_{\beta\eta} A' \quad \Gamma, x : A \vdash B \approx B'}{\Gamma \vdash \lambda x : A.B \approx \lambda x : A'.B'} \qquad \frac{\Gamma \vdash F \approx F' \quad \Gamma \vdash U \approx U'}{\Gamma \vdash FU \approx F'U'}$$

$$\frac{\Gamma \vdash A =_{\beta\eta} A' \quad \Gamma, x : A \vdash B \approx B'}{\Gamma \vdash \Pi x : A.B \approx \Pi x : A'.B'}$$

Lemma 5.1 *The relation $\Gamma \vdash \approx$ is an equivalence relation over well typed terms and if $\Gamma \vdash t \approx t'$, then $\Gamma \vdash t =_{\beta\eta} t'$.*
Proof The only interesting part of the lemma is symmetry which requires an auxilluary lemma stating that if $\Gamma \vdash t \approx t'$ and $\Gamma =_{\beta\eta} \Gamma'$, then $\Gamma' \vdash t \approx t'$. □

Lemma 5.2 *Assume there are typing judgements $\Gamma, x : A, \Delta \vdash t : B$ and $\Gamma \vdash u : A$. If $\Gamma, x : A, \Delta \vdash t \approx t'$ and $\Gamma \vdash u \approx u'$, then $\Gamma, \Delta[u/x] \vdash t[u/x] \approx t'[u'/x]$.*
Proof The proof is by induction on t, using lemma 3.1. □

We would like to apply lemma 2.1 by instantiating R with $\beta/\approx$ and S with $\eta\mathcal{F}/\approx$. However, as lemma 2.5 shows this requires β to be strongly coherent with $\approx$ which is clearly not the case — any β-reduction occurring inside a type abstraction provides a counterexample. What is required is a notion of β-reduction quotiented by type-equivalence, and fortunately this is easily defined:

$$\Gamma \vdash t \Rightarrow_{\beta\approx} t' \text{ iff } \Gamma \vdash t \Rightarrow_{\beta} t' \text{ and } \Gamma \vdash t \not\approx t'$$

Lemma 5.3 *If $\Gamma \vdash (\lambda x : A.t)u$, then there is a reduction $\Gamma \vdash (\lambda x : A.t)u \Rightarrow_{\beta\approx} t[u/x]$*
Proof Assume there is a term $\Gamma \vdash (\lambda x : A.t)u : T$ such that $\Gamma \vdash (\lambda x : A.t)u \approx t[u/x]$ and consider the function E which maps a λC term to an untyped λ-term by erasing all type information in λ and Π-abstractions and which treats sorts and Π's as constants in the untyped calculus. First note that if $\Gamma \vdash t \approx t'$, then $E(t) = E(t')$ and that $E(t[u/x]) = E(t)[E(u)/x]$. Thus we obtain an infinite series of untyped β-reductions

$$E((\lambda x : A.t)u) = (\lambda x.E(t))E(u) \Rightarrow_{\beta} E(t)[E(u)/x] = E(t[u/x]) = E((\lambda x : A.t)u) \Rightarrow_{\beta} \cdots$$

However, the erasure of a typable term is β-SN and hence no such term exists. □

Lemma 5.4 *The relation $\Rightarrow_{\beta\approx}$ is strongly coherent with $\approx$*
Proof Let $\Gamma \vdash t \Rightarrow_{\beta\approx} t'$ and $\Gamma \vdash t \approx u$. The proof is by induction on the term structure of t and we treat only the case where t is of the form $t = FU$. Reductions of proper subterms of t are dealt with by the induction hypothesis, and so the only case left is a reduction $\Gamma \vdash (\lambda x : A.a)b \Rightarrow_{\beta\approx} a[b/x]$. But u must be of the form $(\lambda x : A'.a')b'$ where $\Gamma \vdash b \approx b'$ and $\Gamma, x : A \vdash a \approx a'$. Hence there is a β reduction $\Gamma \vdash (\lambda x : A'.a')b' \Rightarrow_{\beta} a'[b'/x']$ and by lemma 5.3, this is actually a $\Rightarrow_{\beta\approx}$-reduction. Finally, by lemma 5.2, $\Gamma \vdash a[b/x] \approx a'[b'/x]$. □

Lemma 5.5 *The relation $\Rightarrow_{\beta\approx}$ is strongly normalising*
Proof $\Rightarrow_{\beta\approx}$ is strongly normalising because it is a sub relation of $\Rightarrow_{\beta}$. □

Lemma 5.6 *If there is a reduction sequence $\Gamma \vdash t \Rightarrow_{\beta}^{*} t'$, then there is another reduction $\Gamma \vdash t \Rightarrow_{\beta\approx}^{*} t''$ such that $\Gamma \vdash t' \approx t''$.*
Proof The lemma is proved by induction on the length of the reduction sequence $\Gamma \vdash t \Rightarrow_{\beta}^{*} t'$. If the sequence is of length 1, then the lemma holds as either $\Gamma \vdash t \approx t'$, in which case set $t'' = t$, or $\Gamma \vdash t \not\approx t'$, in which case set $t'' = t'$. For

the inductive step:

$$\begin{tikzcd}
t \arrow[r, "\beta^*"] \arrow[dr, "(\beta^{\approx})^*"'] & t' \arrow[r, "\beta"] \arrow[d, "\approx"] \arrow[dr, "(\beta^{\approx})^*"] & u \arrow[d, "\approx"] \\
 & t'' \arrow[dr, "(\beta^{\approx})^*"'] & u'' \arrow[d, "\approx"] \\
 & & t'''
\end{tikzcd}$$

The completions are by induction, the first part of this lemma, and lemma 5.4. □

Lemma 5.7 $\Rightarrow^*_{\beta\approx}$ *satisfies the diamond lemma modulo* $\approx$. *Thus* $\Rightarrow_{\beta\approx} / \approx$ *is confluent.*

Proof Let $\Gamma \vdash t \Rightarrow^*_{\beta\approx} u_i$. By the confluence of $\Rightarrow_\beta$, there is a term u such that $\Gamma \vdash u_i \Rightarrow^*_\beta u$. By lemma 5.6 there are terms v_i such that $\Gamma \vdash u_i \Rightarrow^*_{\beta\approx} v_i$ and such that $\Gamma \vdash v_1 \approx u \approx v_2$. Thus $\Gamma \vdash v_1 \approx v_2$. The second part of the lemma follows from lemma 2.4. □

We now prove $\Rightarrow_{\eta\mathcal{F}} / \approx$ is confluent, and as there is no need to prove $\Rightarrow_{\eta\mathcal{F}} / \approx$ is strongly normalising we show that $\Rightarrow_{\eta\mathcal{F}}$ is coherent with type equivalence.

Lemma 5.8 *The relation* $\Rightarrow_{\eta\mathcal{F}}$ *is coherent with type equivalence.*

Proof Let $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} t'$ and $\Gamma \vdash t \approx u$. The proof is by induction on the rewrite $\Gamma \vdash t \Rightarrow_{\eta\mathcal{R}} t'$ and we treat two cases. If $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \eta^A(t)$, then because t is neutral, so is u. By lemma 5.1, if $\Gamma \vdash t \approx u$, then $\Gamma \vdash t =_{\beta\eta} u$ and so by lemma 3.6 t and u have the same types under Γ. Thus there is an expansion $\Gamma \vdash u \Rightarrow_{\eta\mathcal{F}} \eta^A(u)$. If however, t is of the form $\lambda x : A.t_0$, then for $\Rightarrow_{\eta\mathcal{I}}$-reducts induced by reductions of A, set $u' = u$, while for a reduction $\Gamma \vdash \lambda x : A.t_0 \Rightarrow_{\eta\mathcal{I}} \lambda x : A.t_1$, one applies the induction hypothesis. □

Lemma 5.9 *The relations* $\Rightarrow_{\eta\mathcal{I}} / \approx$ *and* $\Rightarrow_{\eta\mathcal{F}} / \approx$ *are confluent.*

Proof We prove, by induction on term structure, the stronger property that $\Rightarrow_{\eta\mathcal{I}}$ and $\Rightarrow_{\eta\mathcal{F}}$ satisfy the diamond property modulo type-equivalence. We treat the most interesting cases.

If $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \eta^A(t)$ and $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \eta^{A'}(t)$, then by Uniqueness of Products as stated in lemma 3.4, $\Gamma \vdash A =_{\beta\eta} A'$ and hence the two reducts are type equivalent. The inference rule for the type equivalence of Π-types was chosen to allow this step of the proof to go through — see our previous comments. If $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \eta^A(t)$ and $\Gamma \vdash t \Rightarrow_{\eta\mathcal{I}} t'$, then t' must be neutral and by lemma 3.6, in habit the same types under Γ that t does. Thus there are rewrites $\Gamma \vdash t' \Rightarrow_{\eta\mathcal{F}} \eta^A(t')$ and $\Gamma \vdash \eta^A(t) \Rightarrow_{\eta\mathcal{I}} \eta^A(t')$. □

We now establish the final premise of lemma 2.5.

Lemma 5.10 *If $\Gamma \vdash a \Rightarrow_{\beta\approx} b$ and $\Gamma \vdash a \Rightarrow_{\eta\mathcal{I}} t$. Then there are completions of one of the following forms*

$$\begin{tikzcd} a \arrow[rr, "\beta^{\approx}"] \arrow[d, "\eta\mathcal{I}"'] & & b \arrow[d, "\eta\mathcal{I}^*"] \\ t \arrow[r, "(\beta^{\approx})^+"'] & t' \arrow[r, "\approx"'] & c \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, "\beta^{\approx}"] \arrow[d, "\eta\mathcal{I}"'] & & b \arrow[d, "\eta\mathcal{F}"] \\ t \arrow[r, "(\beta^{\approx})^+"'] & t' \arrow[r, "\approx"'] & c \end{tikzcd}$$

If instead $\Gamma \vdash a \Rightarrow_{\eta\mathcal{F}} t$, then the same diagram holds except $\Gamma \vdash b \Rightarrow^*_{\eta\mathcal{F}} c$ is the only possibility.

Proof Notice that by lemma 5.6 and 5.4, we need only show that there is a reduction sequence $\Gamma \vdash t \Rightarrow^+_\beta t'$ containing at least one $\Rightarrow_{\beta\approx}$-reduction. The proof is by induction on the term a and we consider two cases. If $\Gamma \vdash a \Rightarrow_{\eta\mathcal{F}} \eta^A(a)$, then there is no problem in constructing the required completions unless b is of the form $\lambda x : A_0.a_0$. By lemma's 3.4 and 3.3, $\Gamma \vdash A =_{\beta\eta} A_0$ and so we have completions

$$\Gamma \vdash \eta^A(a) \Rightarrow_{\beta\approx} \eta^A(\lambda x : A_0.a_0) \Rightarrow_\beta \lambda x : A.a_0 \approx \lambda x : A_0.a_0 = b$$

Next consider a $\Rightarrow_{\beta\approx}$-rewrite $\Gamma \vdash (\lambda x : A.t)(u) \Rightarrow_{\beta\approx} t[u/x]$. Clearly any rewrite of the subterm A can be completed correctly, while any $\Rightarrow_{\eta\mathcal{I}}$-rewrite induced by rewrites of either t or u can be completed by the usual analysis, e.g. see the substitutivity lemmas of [11]. □

Lemma 5.11 *The relation $\Rightarrow_{\beta\approx}/\approx \cup \eta\mathcal{F}/\approx$ is confluent. Thus $\Rightarrow^*_{\mathcal{F}}$ satisfies the diamond lemma modulo type equivalence.*

Proof By lemmas 5.10 and 2.5, the first relation strongly locally commutes over the later. Thus by lemma 2.1 the two relations commute and since we have also established that each relation is confluent, so is their union. The second part of the lemma is easily proved by unraveling the definitions. □

Confluence modulo type equivalence is sufficient to prove the uniqueness of $\Rightarrow_{\mathcal{F}}$-normal forms.

Lemma 5.12 *Given a judgement $\Gamma \vdash t \approx t'$ where t and t' are $\Rightarrow_{\mathcal{I}}$-normal forms, then $t = t'$. Similarly, given a judgement $\Gamma \vdash t =_{\beta\eta} t'$ where t and t' are $\Rightarrow_{\mathcal{F}}$-normal forms, then $t = t'$.*

Proof The two parts of the lemma are proved simultaneously by induction on the terms t and t'. For example if t is of the form $\lambda x : A.u$, then t' must be of the form $\lambda x : A'.u'$ where $\Gamma \vdash A =_{\beta\eta} A'$ and $\Gamma, x : A \vdash u \approx u'$. Since u and u' are

$\Rightarrow_{\mathcal{I}}$-normal forms, by induction $u = u'$. Similarly as A and A' are $\Rightarrow_{\mathcal{F}}$-normal forms, $A = A'$.

For the second part of the lemma, note that by lemma 5.11, t and t' have type equivalent reducts, but as t and t' are $\Rightarrow_{\mathcal{F}}$-normal forms, t and t' must be type equivalent. Since $\Rightarrow_{\mathcal{F}}$-normal forms are also $\Rightarrow_{\mathcal{I}}$-normal forms, by the first part of this lemma t and t' are equal. $\square$

Given that $\Rightarrow_{\mathcal{F}}$-normal forms are unique, full confluence can be proven by establishing that every term reduces to a normal form. Since every term has a unique β-normal form and by lemma 4.1 $\Rightarrow_{\eta\mathcal{F}}$ preserves β-normal forms, normalisation of $\Rightarrow_{\mathcal{F}}$ is reduced to normalisation of $\Rightarrow_{\eta\mathcal{F}}$. We choose to prove strong normalisation because this is no more difficult.

6 Strong Normalisation for $\Rightarrow_{\eta\mathcal{F}}$

In this section we prove that $\Rightarrow_{\eta\mathcal{F}}$ is strongly normalising, and by lemma 4.1 this implies $\Rightarrow_{\mathcal{F}}$ is normalising. The key to proving $\Rightarrow_{\eta\mathcal{F}}$ is strongly normalising lies in calculating the reducts of a variable. Thus given a typing judgement $\Gamma \vdash T : s$ and a variable $z^s \not\in \mathtt{dom}(\Gamma)$, we define a set of terms $\Delta_z(T)$ as follows

$$\begin{aligned} \Delta_z(\Pi x : A.B) &= \{z\} \cup \{\lambda x : A.v[zu/y] \mid u \in \Delta_x(A) \text{ and } v \in \Delta_y(B)\} \\ \Delta_z(T) &= \{z\} \qquad\qquad\qquad \text{if } T \text{ is not a product} \end{aligned}$$

where the variables x and y are fresh. Note that $\Delta_z(T)$ is always a finite set.

Lemma 6.1 *Let $\Gamma \vdash \mathcal{F}\mathtt{NF}(T) \wedge T : s$. Then $\Gamma, z : T \vdash z \Rightarrow^*_{\eta\mathcal{F}} t$ iff $t \in \Delta_z(T)$.*
Proof Two containments must be established, both by induction over T. The forward inclusion is proved by showing the set $\Delta_z(T)$ is closed under reduction and there are two sub-cases. Firstly if T is not a product, there can be no judgements of the form $\Gamma \vdash z : \Pi x : A.B \wedge \mathcal{F}\mathtt{NF}(\Pi x : A.B)$ as by lemma 3.3 $\Gamma \vdash T =_{\beta\eta} \Pi x : A.B$ and hence by lemmas 5.12 and 4.2, $T = \Pi x : A.B$. Thus z is a normal form. If however T is of the form $\Pi x : A.B$ and $\Gamma \vdash z \Rightarrow_{\eta\mathcal{F}} \eta^{A'}(z)$, then by lemmas 3.3 and 3.4 $A' = A$ and hence $\eta^{A'}(z) \in \Delta_z(T)$. As A is a long $\beta\eta$-normal form, all reducts of $\lambda x : A.v[zu/y]$ must be caused by reductions of either u or v and hence by the induction hypothesis are contained in $\Delta_z(T)$.

For the other containment, assuming T is of the form $\Pi x : A.B$, there is a reduction sequence

$$\Gamma \vdash z \Rightarrow_{\eta\mathcal{F}} \lambda x : A.zx \Rightarrow^*_{\eta\mathcal{F}} \lambda x : A.zu \Rightarrow^*_{\eta\mathcal{F}} \lambda x : A.v[zu/y]$$

where $u \in \Delta_x(A)$ and $v \in \Delta_y(B)$. The last reduction sequence does not break any of the context sensitive restrictions on expansion as zu is not a λ-abstraction. $\square$

A reduction sequence $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{F}} t'$ is called an *t-envelope* iff $t = t'$ or there is a judgement $\Gamma \vdash t : T \wedge \mathcal{F}\mathtt{NF}(T)$ and a term $\alpha \in \Delta_z(T)$ such that $t' = \alpha[t/z]$. As $\Delta_z(T)$ is always finite, there are only a finite number of t-envelopes.

Lemma 6.2 *Any reduction sequence $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{F}} u$ factorises into an internal reduction sequence followed by an envelope, i.e. there is a term t', a reduction sequence $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{I}} t'$ and an envelope $\Gamma \vdash t' \Rightarrow^*_{\eta\mathcal{F}} u$. In addition, the length of the factorised reduction sequence is the same as that of the original.*
Proof The proof is by induction on the length of the sequence $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{F}} u$. If this reduction is of length 0, then the lemma clearly holds, while if $\Gamma \vdash t \Rightarrow_{\eta\mathcal{I}} t_0 \Rightarrow^*_{\eta\mathcal{F}}$, then the lemma holds by induction. The only other possibility is a reduction sequence

$$\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} \lambda x : A.tx \Rightarrow^*_{\eta\mathcal{I}} \lambda x : A.b$$

where there is a reduction sequence $\Gamma, x : A \vdash tx \Rightarrow^*_{\eta\mathcal{F}} b$. By induction this factorises into an internal reduction sequence followed by an envelope

$$\Gamma, x : A \vdash tx \Rightarrow^*_{\eta\mathcal{I}} t'x \Rightarrow^*_{\eta\mathcal{I}} t'u \Rightarrow^*_{\eta\mathcal{F}} b$$

Now the reduction sequence $\Gamma, x : A \vdash t'x \Rightarrow^*_{\eta\mathcal{I}} t'u \Rightarrow^*_{\eta\mathcal{F}} b$ is a $t'x$-envelope and hence the original reduction sequence may be factorised $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{I}} t' \Rightarrow^*_{\eta\mathcal{F}} \lambda x : A.b$. Examination of the proof shows that the length of the original and factorised reduction sequence is the same. □

Lemma 6.3 *There are no infinite sequences of reductions $\Gamma \vdash t \Rightarrow_{\eta\mathcal{F}} t_1 \Rightarrow_{\eta\mathcal{F}} \ldots$*
Proof The proof is by induction on the structure of t and, for a given term, we first show that there are no infinite sequences $\Gamma \vdash t \Rightarrow_{\eta\mathcal{I}} t_1 \Rightarrow_{\eta\mathcal{I}} \ldots$ before obtaining the general result. If t is a variable or sort, then t has no $\Rightarrow_{\eta\mathcal{I}}$-reducts and so is $\Rightarrow_{\eta\mathcal{I}}$-strongly normalising. All $\Rightarrow_{\eta\mathcal{I}}$-reduction sequences of compound terms are induced by $\Rightarrow_{\eta\mathcal{I}}$ or $\Rightarrow_{\eta\mathcal{F}}$-reduction sequences from proper subterms and therefore by the induction hypothesis t is $\Rightarrow_{\eta\mathcal{I}}$-strongly normalising.

Now by lemma 6.2 any reduction sequence $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{F}} u$ factorises into a reduction sequence of the same length $\Gamma \vdash t \Rightarrow^*_{\eta\mathcal{I}} t' \Rightarrow^*_{\eta\mathcal{F}} u$ where the latter reduction is an envelope. As the are no infinite envelopes and t is $\Rightarrow_{\eta\mathcal{I}}$-strongly normalising, t must be $\Rightarrow_{\eta\mathcal{F}}$ strongly normalising. □

Lemma 6.4 *The relation $\Rightarrow_{\mathcal{F}}$ is normalising.*
Proof If t is a term, consider the term t_0 obtained by contracting all β-redexes in t and then performing as many η-expansions as is permissible. Firstly t_0 is well defined as by lemma 6.3 there are no infinite η-expansion sequences. Secondly t_0 is an $\eta\mathcal{F}$-normal form by definition. Thirdly, by lemma 4.1, our restricted η-expansions preserve β-normal forms and hence t_0 is a β-normal form. Hence t_0 is an $\Rightarrow_{\mathcal{F}}$-normal form. □

7 Confluence and Decidability Results

In this final section we prove the main results of this paper, namely confluence of the expansionary rewrite relation, which together with normalisation imply the decidability of $\beta\eta$-equality. Confluence then allows us to prove the decidability of type inhabitation and typability.

Lemma 7.1 *The relation $\Rightarrow_{\mathcal{F}}$ is confluent*

Proof If $\Gamma \vdash t \Rightarrow_{\mathcal{F}} u_1$ and $\Gamma \vdash t \Rightarrow_{\mathcal{F}} u_2$, then by lemma 6.4, there are terms v_1 and v_2 which are $\Rightarrow_{\mathcal{F}}$-normal forms and are also reducts of u_1 and u_2 respectively. As these terms are $\beta\eta$-equal, lemma 5.12 shows that $v_1 = v_2$. Hence $\Rightarrow_{\mathcal{F}}$ is confluent. □

Not only do $\Rightarrow_{\mathcal{F}}$-normal forms exist, but that they are also computable.

Lemma 7.2 *If there is a judgement $\Gamma \vdash t : T$, then for $\mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}$ there are computable terms $\alpha_{\mathcal{R}}$ such that $\Gamma \vdash t \Rightarrow^*_{\mathcal{R}} \alpha_{\mathcal{R}} \wedge \mathcal{R}\mathtt{NF}(\alpha)$.*

Proof We prove the lemma first for the special case where t is a β-normal form and use induction on the well ordering mentioned in lemma 3.7. Since all the immediate subterms of t are β-normal forms, one may apply the induction hypothesis to these immediate subterms to obtain a term t_0 such that $\Gamma \vdash t \Rightarrow^*_{\mathcal{I}} t_0 \wedge \mathcal{I}\mathtt{NF}(t_0)$. If t is a λ-abstraction, then $\Gamma \vdash \mathcal{F}\mathtt{NF}(t_0)$ and so we are done. If not, by lemma 3.7, there is a β-normal form T_0 such that $T_0 < t$ and $\Gamma \vdash T_0 =_{\beta\eta} T$. Applying the induction hypothesis, we obtain a term T_1 such that $\Gamma \vdash T_0 \Rightarrow^*_{\mathcal{F}} T_1 \wedge \mathcal{F}\mathtt{NF}(T_1)$. Now let α be the biggest term in $\Delta_z(T_1)$ (for some appropriate choice of variable). Then $\Gamma \vdash t \Rightarrow^*_{\mathcal{F}} \alpha[t_0/z] \wedge \mathcal{F}\mathtt{NF}(\alpha[t_0/z])$.

Now given an arbitrary judgement $\Gamma \vdash t : T$, there is also a judgement $\Gamma \vdash \beta(t) : T$ where $\beta(t)$ is the β-normal form of t. By the first part of the lemma we may compute the long $\beta\eta$-normal form of $\beta(t)$ which is of course also the long $\beta\eta$-normal form of t. □

Lemma 7.3 *Given a judgement $\Gamma \vdash t : T$, then t is an $\Rightarrow_{\mathcal{F}}$-normal form iff $\Gamma \vdash \mathcal{F}\mathtt{NF}(t)$*

Proof One half of the lemma is proven in lemma 4.2. The second half of the lemma is proven by induction on t while simultaneously proving that if t is an $\Rightarrow_{\mathcal{I}}$-normal form then $\Gamma \vdash \mathcal{I}\mathtt{NF}(t)$. Note that lemma 7.2 is needed to show that if there is a judgement $\Gamma \vdash t : \Pi x : A.B$, then there is another $\Gamma \vdash t : \Pi x : A'.B' \wedge \mathcal{F}\mathtt{NF}(\Pi x : A'.B')$. □

The computability of the long $\beta\eta$-normal form of a term is the key result required to derive the properties of the decidability of type inhabitation and typability.

Lemma 7.4 *Given a context Γ and pre-terms t and T, then it is decidable whether*

- *There is a pre-term T' such that there is a typing judgement $\Gamma \vdash t : T'$*
- *There is a typing judgement $\Gamma \vdash t : T$*

Proof The two parts of the lemma are proved simultaneously by induction firstly on the structure of t. The only difficult part of the analysis is checking whether certain terms are $\beta\eta$-equal and this may be done by constructing their long $\beta\eta$-normal forms as in lemma 7.2 □

8 Conclusions and Further Work

There are several principle avenues for further research. Firstly, it remains an open question as to whether the relation $\Rightarrow_{\mathcal{F}}$, proved weakly normalising here, is actually strongly normalising as the author suspects. A second conjecture is that, under minimal conditions, the methods developed in this paper can be generalised to the more abstract Pure Type Systems. Finally, we wish to make good our promise that η-expansions can open the way to the smooth combination of dependent type systems with algebraic rewrite systems. The research required here is at an advanced stage and a paper on the subject is forthcoming.

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Table 4. Typing and Equality Judgements for λC

Typing Judgements

$$\frac{}{\vdash * : \Box}$$

$$\frac{\Gamma \vdash A : s \quad x^s \not\in \mathtt{dom}(\Gamma)}{\Gamma, x : A \vdash x : A}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash C : s \quad x^s \not\in \mathtt{dom}(\Gamma)}{\Gamma, x : C \vdash A : B}$$

$$\frac{\Gamma \vdash A : B \quad \Gamma \vdash B' : s \quad \Gamma \vdash B =_{\beta\eta} B'}{\Gamma \vdash A : B'}$$

$$\frac{\Gamma \vdash A : s_1 \quad \Gamma, x : A \vdash B : s_2}{\Gamma \vdash \Pi x : A.B : s_2}$$

$$\frac{\Gamma \vdash F : \Pi x : A.B \quad \Gamma \vdash U : A}{\Gamma \vdash FU : B[U/x]}$$

$$\frac{\Gamma, x : A \vdash b : B \quad \Gamma \vdash \Pi x : A.B : s}{\Gamma \vdash \lambda x : A.b : \Pi x : A.B}$$

Equality Judgements

$$\frac{\Gamma \vdash t : C}{\Gamma \vdash t =_{\beta\eta} t}$$

$$\frac{\Gamma \vdash t =_{\beta\eta} t'}{\Gamma \vdash t' =_{\beta\eta} t}$$

$$\frac{\Gamma \vdash t =_{\beta\eta} t' \quad \Gamma \vdash t' =_{\beta\eta} t''}{\Gamma \vdash t =_{\beta\eta} t''}$$

$$\frac{\Gamma \vdash t : \Pi x : A.B}{\Gamma \vdash t =_{\beta\eta} \eta^A(t)}$$

$$\frac{\Gamma \vdash (\lambda x : A.t)u : C}{\Gamma \vdash (\lambda x : A.t)u =_{\beta\eta} t[u/x]}$$

$$\frac{\Gamma, x : A \vdash t =_{\beta\eta} t'}{\Gamma \vdash \lambda x : A.t =_{\beta\eta} \lambda x : A.t'}$$

$$\frac{\Gamma \vdash A =_{\beta\eta} A'}{\Gamma \vdash \lambda x : A.t =_{\beta\eta} \lambda x : A'.t}$$

$$\frac{\Gamma \vdash A =_{\beta\eta} A'}{\Gamma \vdash AB =_{\beta\eta} A'B}$$

$$\frac{\Gamma \vdash B =_{\beta\eta} B'}{\Gamma \vdash AB =_{\beta\eta} AB'}$$

$$\frac{\Gamma \vdash A =_{\beta\eta} A'}{\Gamma \vdash \Pi x : A.B =_{\beta\eta} \Pi x : A'.B}$$

$$\frac{\Gamma, x : A \vdash B =_{\beta\eta} B'}{\Gamma \vdash \Pi x : A.B =_{\beta\eta} \Pi x : A.B'}$$