$\beta\eta$-Equality for Coproducts

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Abstract. Recently several researchers have investigated $\beta\eta$-equality for the simply typed λ-calculus with exponentials, products and unit types. In these works, η-conversion was interpreted as an expansion with syntactic restrictions imposed to prevent the expansion of introduction terms or terms which form the major premise of elimination rules. The resulting rewrite relation was shown confluent and strongly normalising to the long $\beta\eta$-normal forms. Thus reduction to normal form provides a decision procedure for $\beta\eta$-equality.

This paper extends these methods to sum types. Although this extension was originally thought to be straight forward, the proposed η-rule for the sum is substantially more complex than that for the exponent or product and contains features not present in the previous systems. Not only is there a facility for expanding terms of sum type analogous to that for product and exponential, but also the ability to permute the order in which different subterms of sum type are eliminated.

These different aspects of η-conversion for the sum type is reflected in our analysis. The rewrite relation is decomposed into two parts, a strongly normalising and confluent fragment resembling that found in the calculus without coproducts and a relation which generalises the 'commuting conversions' appearing in the literature. This second fragment is proved decidable by constructing for each term its (finite) set of *quasi-normal forms* and, by embedding the whole relation into this conversion relation, decidability, confluence and quasi-normal forms for the full relation are derived.

1 Introduction

Extensional equality for terms of the simply typed λ-calculus requires η-conversion, whose interpretation as a rewrite rule is traditionally as a contraction $\lambda x.fx \Rightarrow f$ where $x \notin \mathtt{FV}(f)$. When combined with the usual β-reduction, the resulting rewrite relation is strongly normalising and confluent, and thus reduction to normal form provides a decision procedure for the associated equality on terms. Unfortunately these properties typically fail if further datatypes are introduced. Even the presence of the unit type (necessary for the definition of types with given constants such as integers and booleans) with η-rewrite rule $t \Rightarrow *$ leads to a loss of confluence [14]. Specifically if f is a variable of type $1 \rightarrow 1$ then the divergence $\lambda x.* \Leftarrow \lambda x.fx \Rightarrow f$ cannot be completed.

Recently several authors [1, 5, 6, 13, 3] have accepted the old proposal [10, 15, 16] that η-conversion be interpreted as an expansion $f \Rightarrow \lambda x.fx$ and the resulting rewrite relation has been shown confluent. In these works infinite reduction sequences such as

$$f \Rightarrow \lambda x.fx \Rightarrow \lambda x.(\lambda y.fy)x \Rightarrow \ldots$$

are prohibited by imposing syntactic restrictions to limit the possibilities for expansion; namely λ-abstractions cannot be expanded and nor can terms which are applied. This restricted expansion relation was shown to be strongly normalising, confluent and to generate the same equational theory as the unrestricted expansionary rewrite relation. Thus $\beta\eta$-equality could be decided by reduction to normal form in the restricted fragment. These normal forms were also shown to be exactly Huet's *long $\beta\eta$-normal forms* [9, 16].

This interpretation of η-conversion as an expansion, and the restrictions required to recover strong normalisation, have a mathematical explanation within categorical models of reduction [12, 17, 18] where types are represented as objects, terms as morphisms and rewrites as 2-cells. In such models introduction and elimination form an adjoint pair of functors whose local unit and counit [11, 8] correspond to η-expansion and β-contraction respectively. These are linked by local triangle laws which when cut give rise to the restrictions mentioned above.

This paper uses the same categorical methods to derive rewrite rules for coproducts and analyses the resulting rewrite relation. Although originally thought to be a straightforward extension of previous results, the η-rule for coproducts turns out to be substantially more complex than that for products and exponentials. Not only is there a facility for expanding terms of sum type analogous to that for expanding terms of product and exponential type, but also the ability to permute the order in which different subterms of sum type may be eliminated.

These different aspects of η-conversion for the sum type are reflected in our analysis. After defining the terms of the calculus, the categorical methods outlined above are used to define an extensional rewrite relation over these terms, and this relation is then decomposed into two fragments. The first part of this decomposition contains β-redexes, commuting conversions and limited possibilities for η-expansion and is strongly normalising and confluent. All subterms of a normal form are either of base type, the major premise of an elimination rule or a *quasi-introduction term* and so normal forms of this relation may be seen as generalising the long $\beta\eta$-normal forms to this calculus.

The second part of the decomposition allows permutation of the order in which subterms of sum type may be eliminated, an example of which are the 'commuting conversions' appearing in the literature [16, 7]. For each term there are a finite set of such permutations, and so in general unique normal forms do not exist for this relation. Instead to each term is associated a (finite) set of *quasi-normal forms* and terms equivalent in the associated equational theory are shown to have the same set of quasi-normal forms. Confluence and decidability of this conversion relation are corollaries to these results. Finally by appropriately embedding the whole relation in the conversion relation, full confluence and decidability may be proved.

Historically the use of expansions for products and exponentials can be traced back to [15], although the proof that they form a strongly normalising relation had to wait a decade for the papers mentioned above. A partial solution to the problem of $\beta\eta$-equality for coproducts was provided by [6] but in this approach confluence can only be proved for terms of ground type. As of writing the author is aware of several other on-going attempts to tackle this problem, but is unaware of any actual solutions.

The rest of this paper is organised as follows. Section 2 contains notation required later, section 3 a definition of the term calculus and section 4 uses categorical methods to derive an expansionary η- and contractive β-rewrite rule for the coproduct. Section 5 defines the conversion relation and proves decidability and confluence for this fragment, while section 6 deals with the strongly normalising fragment and its embedding into the conversion relation. The paper concludes with some remarks on the possible direction of future research.

2 Notation

While basic knowledge of term rewriting is assumed [4, 10], an introduction to *occurrences* is given (a full development of which may be found in [10]). Occurrences are sequences of natural numbers used to index the subterms of a term and whose analysis forms the technical core of this paper. Let $\mathcal{N}^*$ be the set of sequences of natural numbers with the empty sequence denoted ϵ, while $u.v$ denotes the concatenation of u with v. If $u \neq \epsilon$, then u^+ is the sequence obtained by omitting the last element of u, while u^- is the sequence obtained by omitting the first element. The prefix partial ordering is defined $u \leq v$ iff $\exists w.v = u.w$ and then $v/u = w$. These operations on sequences are extended pointwise to sets of sequences e.g. $X/u = \{w | u.w \in X\}$.

Now let $\mathcal{T}$ be the terms of some calculus. Given any $t \in \mathcal{T}$, its set of *occurrences* $\mathtt{O}\ (t) \subseteq \mathcal{N}^*$, and the subterm indexed at occurrence $\sigma \in \mathtt{O}\ (t)$, denoted t/σ, is defined

- If $t = x$, where x is a variable, then $\mathtt{O}\ (t) = \{\epsilon\}$ and $t/\epsilon = t$
- If $t = Ft_0 \ldots t_n$, then $\mathtt{O}\ (t) = \{\epsilon\} \cup \{i.\sigma | i \leq n, \sigma \in \mathtt{O}\ (t_i)\}$ and

$$t/\sigma = \begin{cases} t & \text{if } \sigma = \epsilon \\ t_i/\sigma^- & \text{if } \sigma \neq \epsilon \text{ and } \sigma_0 = i \end{cases}$$

When no danger of confusion exists, the distinction between an occurrence and the subterm so indexed may be blurred. The conversion relation is not left linear, i.e. different occurrences in a redex, indexing syntactically equal subterms, may be mapped to the same occurrence in the reduct. Thus define a set X of occurrences to be *consistent* iff $\forall \sigma, \sigma' \in X.t/\sigma = t/\sigma'$ and if X is non-empty, the subterm(s) so indexed are denoted t/X. Finally $t[\sigma_i \leftarrow u_i]_{i \in \mathcal{I}}$ denoted the textual replacement of terms u_i at occurrences σ_i.

The conversion relation, although not strongly normalising, does posses slightly weaker properties which are nevertheless of both theoretical and practical importance. A *quasi-normal* form for a term t is a term t' such that all reducts of t reduce to t' and the set of R-quasi-normal forms of t is denoted $R(t)$. The one-step reducts of a relation R are denoted $x/R = \{x' | (x, x') \in R\}$, the reflexive closure R^+ and the reflexive transitive closure R^*. If R is an equivalence relation, the equivalence class of an element is denoted $x^=$, while if a term t is R-strongly normalising, its R-rank is denoted $|t|_R$. Finally term constructors are often considered operators on sets of terms, i.e. $F(S_1, \ldots, S_n)$ denotes the set of terms $\{F(t_1, \ldots, t_n) | t_i \in S_i\}$.

3 Almost Bicartesian Closed Logic

Although this paper is primarily concerned with the definition and decidability of $\beta\eta$-equality for coproducts, in order to maintain continuity with previous work and to avoid certain trivial simplifications, a calculus which includes products, terminal object and exponentials is studied. This calculus is called "Almost Bicartesian Closed" as it corresponds to the internal language of bicartesian closed categories, without an initial object. Although the details have not been fully verified, it is hoped that the techniques developed here are sufficient to cope with the addition of an initial object.

The *types* of "Almost Bicartesian Logic", denoted BCC^-, are freely generated by the syntax

$$T := B \mid 1 \mid T + T \mid T \to T \mid T \times T$$

where B is any base type. For each type T, there are constants $Con(T)$ including the special constant $* \in Con(1)$. Now let Var be a set of variables disjoint from the constants. A context is a list of pairs of variables and types, written $x_1 : \sigma_1, \ldots, x_n : \sigma_n$ such that the variables are pairwise distinct. The concatenation of contexts Γ and Δ is written Γ, Δ. The *term judgements* of BCC^- are of the form $\Gamma \vdash t : T$ and are generated by the traditional structural rules of *Weakening, Contraction* and *Exchange* and by the following logical rules

$$\frac{x \in Var}{x : A \vdash x : A}\ Ax \qquad \frac{c \in Con(A)}{\vdash c : A}\ Cons$$

$$\frac{\Gamma \vdash e : A \quad \Gamma \vdash e' : B}{\Gamma \vdash \langle e, e' \rangle : A \times B}\ \times R \qquad \frac{\Gamma \vdash t : A_1 \times A_2}{\Gamma \vdash \pi_i t : A_i}\ \times L$$

$$\frac{\Gamma \vdash t : A_i}{\Gamma \vdash \mathbf{in}_i(t) : A_1 + A_2}\ +R \qquad \frac{\Delta \vdash t : A + B \quad \begin{array}{c}\Gamma, x : A \vdash u : C \\ \Gamma, y : B \vdash v : C\end{array}}{\Gamma, \Delta \vdash \mathbf{case}(t, x.u, y.v) : C}\ +L$$

$$\frac{\Gamma, x : A \vdash e : B}{\Gamma \vdash \lambda x.e : A \to B}\ \to R \qquad \frac{\Gamma \vdash e : A \to B \quad \Delta \vdash e' : A}{\Gamma, \Delta \vdash ee' : B}\ \to L$$

where the variables occuring in Γ and Δ must be disjoint in those rules involving multiple premises. Given any term judgement $\Gamma \vdash t : T$, we say t is a term of type T and this is written $t : T$. The free variables of a term t are denoted $\mathtt{FV}(t)$ and substitution of terms for free variables of the same type is defined as expected. The suggested η-rewrite rule for coproducts considers terms expressed as substitutions, i.e. subterms whose free variables are not bound in their context. Thus we define the variables bound at an occurrence $\sigma \in \mathtt{O}\ (t)$ as

$$\mathtt{BV}(\sigma, t) = \begin{cases} \emptyset & \text{if } \sigma = \epsilon \\ \{x\} \cup \mathtt{BV}(\sigma^-, t') & \text{if } t = \lambda x.t' \text{ and } \sigma \neq \epsilon \\ \{x_1\} \cup \mathtt{BV}(\sigma^-, t') & \text{if } t = \mathbf{case}(u, x_1.v_1, x_2.v_2) \text{ and } \sigma \geq 1 \\ \{x_2\} \cup \mathtt{BV}(\sigma^-, t') & \text{if } t = \mathbf{case}(u, x_1.v_1, x_2.v_2) \text{ and } \sigma \geq 2 \\ \mathtt{BV}(\sigma^-, t/\sigma_0) & \text{otherwise} \end{cases}$$

and then the free occurrences of a term by $\mathtt{FO}(t) = \{\sigma \in \mathtt{O}\ (t) | \mathtt{FV}(t/\sigma) \cap \mathtt{BV}(\sigma, t) = \emptyset\}$

Familiarity with calculi such as that above is assumed [7, 2]. A term e is called an *introduction* term if it is a λ-abstraction, pair, injection or the constant $*$. If a term is not an introduction term, then it is called an *elimination* term. An occurrence $\sigma \in \mathtt{O}\ (t)$ is the *major premise* of an elimination rule if the subterm so indexed is either applied, projected or the first argument of a *case*-expression.

4 A Rewrite Relation for BCC^-

In [13] extensional rewrite relations for the product, unit and exponential were derived by constructing categorical models of reduction and taking introduction and elimination to be (locally) adjoint functors. When applied to coproducts this approach again generates a contractive β-rewrite rule and an expansionary η-rewrite rule. If $\mathcal{C}(\Gamma, X)$ is the category whose objects are judgements of the form $\Gamma \vdash e : X$ and whose morphisms are rewrites between appropriate judgements, then coproduct introduction and elimination are both functors between the categories shown. These functors constitute an adjoint pair

$$\mathcal{C}(\Gamma.x : A, C) \times \mathcal{C}(\Gamma.y : B, C) \underset{(\_[\mathbf{in}_1(x)/z], \_[\mathbf{in}_2(y)/z])}{\overset{\mathbf{case}(z, x.\_, y.\_)}{\rightleftarrows}} \mathcal{C}(\Gamma.z : A + B, C) \qquad \top$$

and the associated unit and co-unit are the rewrite rules.

$$\begin{array}{ll} (\beta_{+,1}) & \mathbf{case}(\mathbf{in}_1(x), x.u, y.v) \Rightarrow u \\ (\beta_{+,2}) & \mathbf{case}(\mathbf{in}_2(x), x.u, y.v) \Rightarrow v \\ (\eta_+) & e \Rightarrow \mathbf{case}(z, x.e[\mathbf{in}_1(x)/z], y.e[\mathbf{in}_2(y)/z]) \end{array}$$

These reduction rules, when closed under substitution and taken together with the reduction rules for the exponential, product and unit connectives, generate *expansionary rewrite relation*, denoted $\Rightarrow$, whose analysis is the subject of the rest of the paper.

$$\begin{array}{lrl} (\beta_{\times,1}) & \pi_0\langle a, b\rangle & \Rightarrow a \\ (\beta_{\times,2}) & \pi_1\langle a, b\rangle & \Rightarrow b \\ (\eta_\times) & c & \Rightarrow \langle \pi_0 c, \pi_1 c\rangle \\ (\beta_\rightarrow) & (\lambda x.t)u & \Rightarrow t[u/x] \\ (\eta_\rightarrow) & t & \Rightarrow \lambda x.tx \\ (\eta_1) & a & \Rightarrow * \\ (\beta_{+,1}) & \mathbf{case}(\mathbf{in}_1(t), x.u, y.v) & \Rightarrow u[t/x] \\ (\beta_{+,2}) & \mathbf{case}(\mathbf{in}_2(t), x.u, y.v) & \Rightarrow v[t/y] \\ (\eta_+) & e[e'/z] & \Rightarrow \mathbf{case}(e', x.e[\mathbf{in}_1(x)/z], y.e[\mathbf{in}_2(y)/z]) \end{array}$$

where various restrictions are implicitly imposed to ensure well-typedness. In addition, to avoid the capture of free variables, $x, y \notin \mathtt{FV}(e)$ in the rewrite rule η_+ and in $\eta_\rightarrow$ we assume $x \notin \mathtt{FV}(t)$.

The rewrite rule η_+ is highly non-local in that (free) subterms of sum type may be expanded to the head of the term and as a result the rewrite rule is significantly more complex than $\eta_\times$ and $\eta_\rightarrow$. As terms typically contain many such subterms a unique normal form cannot be given; rather we may associate to each term a set of normal forms, one for each of the different permutations in which subterms may be expanded. For example the term $\langle \mathbf{case}(t, x.x, y.y), \mathbf{case}(t', x'.x', y'.y')\rangle$ has two normal forms

$$\mathbf{case}(t, x.\mathbf{case}(t', x'.\langle x, x'\rangle, y'.\langle x, y'\rangle), y.\mathbf{case}(t', x'.\langle y, x'\rangle, y'.\langle y, y'\rangle))$$

and

$$\mathbf{case}(t', x'.\mathbf{case}(t, x.\langle x', x\rangle, y.\langle x', y\rangle), y'.\mathbf{case}(t, x.\langle y', x\rangle, y.\langle y', y\rangle))$$

depending on the order in which subterms are expanded. To accommodate this feature, the rewrite rule η_+ is decomposed into two parts, the first of which converts subterms of sum type into the major premises of *case*-expressions and the second which permutes the order in which such subterms are eliminated.

A special case of η_+ occasionally mentioned in the literature [7, 16] is the rewrite rule

$$t \Rightarrow \mathbf{case}(t, x.\mathbf{in}_1(x), y.\mathbf{in}_2(y))$$

This specialisation of η_+ is more akin to $\eta_\times$ and $\eta_\rightarrow$ and indeed, once suitable restrictions have been imposed upon the applicability of these expansions, and taken together with the β-redexes and commuting conversions, forms a strongly normalising and confluent relation. The proof of normalisation is essentially an adaptation of that in [13], although a couple of new innovations are required to deal with the non-congruent nature of this relation. The normal forms of this relation generalise the notion of a long $\beta\eta$-normal form to this calculus as all subterms of a normal form are either of base type, the major premise of an elimination rule or a *quasi-introduction term*. As with the calculus without coproducts, these $\beta\eta$-normal forms can be calculated by first contracting all β-redexes and then performing all remaining expansion possibilities.

The second part of the decomposition is a generalisation of the 'commuting conversions' appearing in [16, 7]. A *conversion* is a subterm occuring as the major premise of a sum elimination and the *conversion relation* develops an algebra of these conversions, allowing them to be identified, discarded or expanded to the head of a term, e.g. the two normal forms above are interconvertable in the conversion relation. Although not strongly normalising, each term has a (finite, enumerable) set of quasi-normal forms and terms equivalent in the equational theory generated by the conversion relation have the same set of quasi-normal forms. Confluence and decidability of the conversion relation are corollaries to these results. The conversion relation is fairly complex and the mathematics required for its complete analysis is too detailed to include in full here. However we indicate the intuitions and motivations behind our definitions and theorems.

Finally the whole expansionary rewrite relation is shown confluent and decidable by embedding it into the conversion relation. Formally terms equivalent in the full theory have normal forms under the first part of the decomposition which are equivalent in the conversion relation.

5 The Conversion Relation

The η_+-rewrite rule extracts free subterms of sum type and inserts injections at their occurrences in the redex; if the original subterm was the major premise of a *case*-expression new β_+-redexes are created by this process. The conversion relation restricts application of the η_+-rewrite rule to extract only those (free) subterms which occur as the major premise of a *case*-expression, and then contracts the resulting β-redexes.

Given a term t, its set of *conversions* is defined by

$$\mathtt{C}\ (t) = \{\sigma \in \mathtt{O}\ (t) | \sigma \text{ occurs as the first argument of a } \textit{case}\text{-expression }\}$$

The free conversions of t are simply those occurrences which are both free and conversions $\mathtt{FC}(t) = \mathtt{C}\ (t) \cap \mathtt{FO}(t)$. Every conversion has a *binding* which consists of the variables bound at the *case*-expression associated to the conversion. These variable bindings play an important role in avoiding variable capture but due to lack of space we often gloss over the details of their treatment. We also assume that whenever sets of conversions are considered, the subterms so indexed are all of the same type and have the same binding.

If $X \subseteq \mathtt{C}\ (t)$, then the result of contracting the β-redexes formed upon insertion of injections at these occurrences are called the *first and second residues* and may be calculated

$$t \setminus_i X = \begin{cases} t & \text{if } X = \emptyset \\ v_i \setminus_i X_i & \text{if } 0 \in X \text{ and } t = \mathbf{case}(u, x.v_1, y.v_2) \\ F(t_1 \setminus_i X_1, \cdots, t_n \setminus_i X_n) & \text{if } X \neq \emptyset,\ 0 \notin X \text{ and } t = F(t_1, \cdots, t_n) \end{cases}$$

where $i = 1, 2$ and $X_n = X/n$.

Lemma 5.1 *Given a set of conversions $X \subseteq \mathtt{C}\ (t)$ binding the variables x_1 and x_2, then for $i = 1$ or 2 there is a reduction sequence $t[\sigma \leftarrow \mathbf{in}_i(x_i)]_{\sigma \in X} \Rightarrow^* t \setminus_i X$.*
Proof A straight forward induction over t □

Residues form part of the definition of the conversion relation which is formally given in terms of a calculus for deriving triples of the form $\langle \sigma, X \rangle : t \Rightarrow_c t'$ where σ is the depth in the term where the rewrite occurs and X are conversions to be expanded, i.e. $X \subseteq \mathtt{FC}(t/\sigma)$. These triples are generated by three inference rules.

$$\text{Expansion}\ \frac{X \subseteq \mathtt{FC}(t) \quad X \text{ consistent} \quad X \neq \emptyset}{\langle \epsilon, X \rangle : t \Rightarrow_c \mathbf{case}(t/X, x.t \setminus_1 X, y.t \setminus_2 X)}$$

$$\text{Weakening}\ \frac{x \notin FV(t) \quad y \notin FV(t)}{\langle \epsilon, \emptyset \rangle : \mathbf{case}(u, x.t, y.t) \Rightarrow_c t}$$

$$\text{Congruence}\ \frac{\langle \sigma, X \rangle : t_j \Rightarrow_c t'_j}{\langle j.\sigma, X \rangle : \mathcal{T}(t_0, \ldots, t_n) \Rightarrow_c \mathcal{T}(t_0, \ldots, t_n)[j \leftarrow t'_j]}$$

where in the first clause x, y are the variables bound by each $\sigma \in X$ and to avoid variable capture $x, y \notin \mathtt{FV}(t) \cup \mathtt{BV}(\sigma, t)$. These conditions can always be met, if necessary by a change of bound variables. The label part of a rewrite is sometimes omitted.

The *Expansion* clause requires the set X of conversions to be free and consistent so that the redex may be expressed in a form compatible with the η_+-rewrite rule. In addition this set is required to be non-empty to prevent expansions of the form $u \Rightarrow \mathbf{case}(t, x.u, y.u)$ which would allow terms to grow arbitrary large, new free variables to be introduced and other undesirable features. However to ensure these terms are identified in the equational theory generated by the conversion relation, rewrites of this form have been inverted and included under the *Weakening* clause.

Lemma 5.2 *Given a triple $\langle \sigma, X \rangle : t \Rightarrow_c t'$, then $t = t'$ in the expansionary rewrite relation.*
Proof Assume first that $\sigma = \epsilon$. If the rewrite is of the form $\mathbf{case}(u, x.t, y.t) \Rightarrow_c t$ then given a variable z not free in t

$$t \ = \ t[u/z] \Rightarrow_{\eta_+} \mathbf{case}(u, x.t[\mathbf{in}_1(x)/z], y.t[\mathbf{in}_2(y)/z]) \ = \ \mathbf{case}(u, x.t, y.t)$$

If however X in non-empty then, since X is non-empty and consists of free consistent conversions,

$$\begin{aligned} t \ &= \ t[\sigma \leftarrow z]_{\sigma \in X}\,[z := t/X] \\ &\Rightarrow_{\eta_+} \mathbf{case}(t/X, x.t[\sigma \leftarrow \mathbf{in}_1(x)]_{\sigma \in X}, y.t[\sigma \leftarrow \mathbf{in}_2(y)]_{\sigma \in X}) \\ &\Rightarrow^* \mathbf{case}(t/X, x.t \setminus_1 X, y.t \setminus_2 X) \end{aligned}$$

Finally if $\sigma \neq \epsilon$ then, as both relations are congruences, the lemma follows by induction. □

The conversion relation is so named because they generalise the *commuting conversions* [7, 16], i.e. redexes formed when *case*-expressions form the major premise of an elimination rule. The reader is invited to check the following commuting conversion (which illustrates many interesting features and is used later) may be derived as a conversion rewrite with label $\langle \epsilon, \{00\} \rangle$.

$$\begin{array}{ll} \beta_{+,+}: & \mathbf{case}(\mathbf{case}(t, x.u, y.v), x'.u', y'.v') \Rightarrow \\ & \mathbf{case}(t, x.\mathbf{case}(u, x'.u', y'.v'), y.\mathbf{case}(v, x'.u', y'.v')) \end{array} \tag{1}$$

The core of our analysis of the conversion relation is to use the structure of a rewrite, represented in the associated label, to describe its action on arbitrary conversions. More formally to each rewrite $r : t \Rightarrow t'$ is associated a function $\overline{r} : \mathtt{C}\ (t) \rightarrow \mathcal{P}\mathtt{C}\ (t')$ which relates conversions in the redex, called *ancestors*, to conversions in the reduct, called *descendants*. A single conversion may have more than one descendant and the ordering on conversions is not necessarily preserved, e.g. in reduction 1 the conversion $00 \mapsto \{0\}$ while $0 \mapsto \{10, 20\}$. Another interesting reduction is

$$\langle \epsilon, \{0, 10\} \rangle : \mathbf{case}(t, x.\mathbf{case}(t, x.u, y.v), y.s) \Rightarrow \mathbf{case}(t, x.u, y.s) \tag{2}$$

which shows how a conversion, e.g. any inside v, may have no descendants, and how a conversion in the reduct may have more than one ancestor. The construction of $\overline{r}$ is fairly lengthy and involved and so only a sketch is given. Firstly given (consistent) conversions X the set $\mathtt{C}\ (t)$ may be partitioned into those conversions which are sub-conversions of (unique) members of X, those conversions which have descendants in one or both of the residues and those conversions which fit into neither of these categories and hence have no descendants. If the calculation of a residue is viewed as indicating a kind of path through the term, then conversions of this last type are those which are unreachable, e.g. conversions like those inside v in reduction 2. The unique descendant of a conversion σ in a residue $t \setminus_i X$ is given by the following function, which is undefined if no such descendant exists.

$$\Omega_i(X, \sigma) = \begin{cases} \Omega_i(X/i, \sigma^-) & \text{if } 0 \in X \text{ and } \sigma \geq i \\ \text{undefined} & \text{if } 0 \in X \text{ and } \sigma \not\geq i \\ 0 & \text{if } 0 \notin X \text{ and } \sigma = 0 \\ \sigma_0.\Omega_i(X/\sigma_0, \sigma^-) & \text{otherwise} \end{cases}$$

The subterm indexed by descendant conversions is given by $(t \setminus_i X)/\Omega_i(X, \tau) = (t/\tau) \setminus_i (X/\tau)$. Now if τ a conversion and r is a *Weakening*, its descendents are

$$\overline{r}(\tau) = \begin{cases} \emptyset & \text{if } \tau \geq 0 \\ \{\tau^-\} & \text{otherwise} \end{cases}$$

while if r is an expansion of the non-empty set of conversions X

$$\overline{r}(\tau) = \begin{cases} \{0.\tau/\sigma\} & \text{if there is a } \sigma \in X \text{ with } \tau \geq \sigma \\ \{1.\Omega_1(X, \tau), 2.\Omega_2(X, \tau)\} & \text{otherwise} \end{cases}$$

where σ in the first clause is necessarily unique and those functions undefined in the second clause are deleted. Finally if r is induced by a congruence then $\overline{r}$ is calculated by induction. The functions $\overline{r}$ are extended pointwise to sets of conversions. Rather lengthy, but tedious, inductive arguments show that $\overline{r}(\tau)$ actually consists of conversions, and that $\overline{r}$ is surjective, i.e. the the function

$$r^{-1}(\tau) = \{\sigma \in \mathtt{C}\ (t) | \tau \in \overline{r}(\sigma)\}$$

does not return the empty set, i.e. the conversion relation does not "create" new conversions. This surjectivity result means that the possibilities for further reduction in a reduct may be traced back to the associated redex, i.e. a static analysis of all the possibilities for reduction may be given. In the next two subsections these comments are validated by the construction of the normal forms of a given term.

5.1 A Decidability Result

Given a rewrite $r : t \Rightarrow_c t'$ and a set of conversions $X \subseteq \mathtt{C}\ (t)$, we shall try to find conditions under which r may be decomposed into its action on individual conversions and its action on the residues, i.e. find conditions under which rewrites of the following form exist

$$r/\sigma' : t/\sigma \Rightarrow_c t'/\sigma' \quad \text{and} \quad r \setminus_i X : t \setminus_i X \Rightarrow_c t' \setminus_i \overline{r}(X)$$

where $\sigma \in X$ and $\sigma' \in \overline{r}(\sigma)$. Such decompositions of a rewrite into rewrites on smaller terms will lead to a recursive characterisation of the possibilities of rewriting for a given term and thus eventually to confluence and decidability. The reduction 1 shows that in general a rewrite may not be localised to a conversion as the conversion 0 is mapped to the conversions 10 and 20 but no reduction exists between the corresponding subterms. This problem occurs as the conversion 0 is mapped into the residues while one of its sub-conversions is expanded to the head of the term and is thus 'removed' from the original conversion. The prefix-ordering endows conversions with a layer structure and the key to localising a rewrite to a conversion is to ensure that this layer structure is preserved. Thus a rewrite $\langle \tau, X \rangle : t \Rightarrow t'$ is said to *preserve* a conversion $\sigma \in \mathtt{C}\ (t)$ iff $\forall \omega \in X. \neg(\tau < \sigma < \omega)$, i.e. no subconversions of σ are expanded outside of σ. This generates a subrelation where all conversions are preserved.

$$\Rightarrow_p = \{r : t \Rightarrow t' | r \text{ preserves } \mathtt{C}\ (t)\}$$

Note the commuting conversion 1 represents the breakdown of the conversion layer structure, i.e. any $\Rightarrow_c$-reduction may be expressed as a sequence of $\Rightarrow_p$-reductions and $\Rightarrow_{\beta_+,+}$-reductions.

Given a rewrite r, a set of conversions X is *r-closed* iff $r^{-1}\overline{r}(X) = X$. Such sets of conversions induce rewrite in the associated residues, the closure requirement is linked to the non-left linearity of the conversion relation. There is also a slight complication regarding the direction of the residual rewrite. For example if r represents a basic expansion of conversions $Y \subseteq \mathtt{C}\ (t)$, then the obvious candidate for $r \setminus_i X$ is the basic expansion of the descendants of Y in the residue $t \setminus_i X$. However this set may be empty, in which case the direction of the rewrite must be reversed.

Lemma 5.3 *Let $r : t \Rightarrow_c t'$ preserve $\sigma \in \mathtt{C}\ (t)$ and $\sigma' \in \overline{r}(\sigma)$. Then there is a rewrite $r/\sigma' : t/\sigma \Rightarrow t'/\sigma'$ and if X is an r-closed set of conversions then there are either rewrites $r \setminus_i X : t \setminus_i X \Rightarrow t' \setminus_i \overline{r}(X)$ or rewrites in the reverse direction. In addition if r preserves all conversions then so do the localised and residual rewrites.*

Proof The proof is by induction on r. □

We can now prove the conversion relation decidable but, because of the conditions required by the above lemma, this is done in two stages, firstly for the subrelation $\Rightarrow_p$ and then for the whole conversion relation. Decidability of $\Rightarrow_p$ is proved by constructing for each term its finite and enumerable set of $\Rightarrow_p$-quasi normal forms and using Lemma 5.3 to show that $\Rightarrow_p$-equivalent terms have the same set of quasi-normal forms.

If a term is a quasi-normal form containing free conversions, then the term must be a *case*-expression, as otherwise a rewrite to such a term would exist, but not one in the other direction. Thus the construction of normal forms is essentially a process of expanding as many conversions as possible. However as a non-free conversion may have a free descendant, we must consider not just free conversions but also *potentially free conversions*. Secondly as these normal forms are to be $\Rightarrow_p$-reducts, only *maximal* conversions are to be expanded, i.e. those conversions which are not subconversions of other conversions. The construction also performs two other tasks, checking for possible applications of *Weakening* and also ensuring as much identification of conversions as possible.

The set of *maximal conversions* is defined $\mathtt{MC}(t) = \{\sigma \in \mathtt{C}\ (t) | \not\exists \sigma' \in \mathtt{C}\ (t). \sigma' < \sigma\}$. The construction of quasi-normal forms is presented in terms of an inductively defined relation $\mathtt{NF}_p$

- If t is a variable then $\mathtt{NF}_p(t) = \{t\}$.

- If t is not a variable then define the set of *potentially free conversions*, denoted $\mathtt{PFC}\ (t)$, as

$$\{\sigma \in \mathtt{C}\ (t) | \forall u \in \mathtt{NF}_p(t/\sigma).\mathtt{BV}(\sigma) \cap \mathtt{FV}(u) = \emptyset\}$$

  The set of *maximal potentially free conversions* of t, denoted $\mathtt{MPFC}(t)$, consists of those potentially free conversions which are also maximal and this set is equipped with an equivalence relation $\sigma_1 \sim \sigma_2$ iff $\mathtt{NF}_p(t/\sigma_1) = \mathtt{NF}_p(t/\sigma_2)$. There are three inference rules for constructing normal forms

$$\frac{\mathtt{MPFC}(t) = \emptyset \quad t_i \Rightarrow_{\mathtt{NF}_p} \alpha_i}{\mathcal{T}(t_1, \ldots, t_n) \Rightarrow_{\mathtt{NF}_p} \mathcal{T}(\alpha_1, \ldots, \alpha_n)}$$

$$\frac{\sigma \in \mathtt{MPFC}(t) \quad \mathtt{NF}_p(t \setminus_1 \sigma^{=}) = \mathtt{NF}_p(t \setminus_2 \sigma^{=}) \quad t \setminus_1 \sigma^{=} \Rightarrow_{\mathtt{NF}_p} u}{t \Rightarrow_{\mathtt{NF}_p} u}$$

$$\frac{\sigma \in \mathtt{MPFC}(t) \quad \mathtt{NF}_p(t \setminus_1 \sigma^{=}) \neq \mathtt{NF}_p(t \setminus_2 \sigma^{=}) \quad t \setminus_i \sigma^{=} \Rightarrow_{\mathtt{NF}_p} \beta_i \quad t/\sigma \Rightarrow_{\mathtt{NF}_p} \alpha}{t \Rightarrow_{\mathtt{NF}_p} \mathbf{case}(\alpha, x.\beta_1, y.\beta_2)}$$

where in the last clause x, y are the variables bound by the set of conversions $\sigma^{=}$. The set of terms $\mathtt{NF}_p(t)$ is clearly non-empty, finite, enumerable and the maximality condition ensures that if $t' \in \mathtt{NF}_p(t)$ then t' is a $\Rightarrow_p$-reduct of t.

Proposition 5.4 *Let* $r : t \Rightarrow_p t'$.

- *If* $\sigma \in \mathtt{C}\ (t)$ *and* $\sigma' \in \overline{r}(\sigma)$*, then* $\sigma \in \mathtt{MPFC}(t)$ *iff* $\sigma' \in \mathtt{MPFC}(t')$ *and for such a* σ*,* $\overline{r}(\sigma^{=}) = \sigma'^{=}$*.*
- *The sets* $\mathtt{NF}_p(t) = \mathtt{NF}_p(t')$ *are equal*

Proof The lemma is proved simultaneously by induction on the sum of the sizes of the terms in question.

- That σ is maximal iff σ' is follows by induction on the definition of the function $\overline{r}$. Secondly, by induction $\mathtt{NF}_p(t/\sigma) = \mathtt{NF}_p(t'/\sigma')$ and so for any $u \in \mathtt{NF}_p(t/\sigma)$, there is a reduction $t \Rightarrow t' \Rightarrow t'[\sigma' \leftarrow u]$. We may further show that $\mathtt{BV}(\sigma) \cap \mathtt{FV}(u) = \mathtt{BV}(\sigma') \cap \mathtt{FV}(u)$ and so σ is potentially free iff σ' is. Finally given $\tau' \in (\sigma')^{=}$, there is a $\tau \in \mathtt{MPFC}(t)$ such that $\tau' \in \overline{r}(\tau)$ and by induction
$$\mathtt{NF}_p(t/\tau) = \mathtt{NF}_p(t'/\tau') = \mathtt{NF}_p(t'/\sigma') = \mathtt{NF}_p(t/\sigma)$$
  Thus $\tau \in \sigma^{=}$ and $\tau' \in \overline{r}(\sigma^{=})$. Similar arguments prove the reverse containment.
- There are two possibilities. Firstly $\mathtt{MPFC}(t) = \emptyset$ then by the first part of this lemma $\mathtt{MPFC}(t') = \emptyset$ and so the result follows by induction. If however $\exists \sigma \in \mathtt{MPFC}(t)$ then $\overline{r}(\sigma) \subseteq \mathtt{MPFC}(t')$. As $\sigma^{=}$ is r-closed, there are rewrites
$$t/\sigma \Rightarrow t'/\sigma' \quad \text{and} \quad t \setminus_1 \sigma^{=} \Rightarrow t' \setminus_1 \overline{r}(\sigma^{=}) \quad \text{and} \quad t \setminus_2 \sigma^{=} \Rightarrow t' \setminus_2 \overline{r}(\sigma^{=})$$
  where $\sigma' \in \overline{r}(\sigma)$ and the second and third reductions may be reversed. For each of these equalities the set of normal forms of the left hand side is the same as those of the right hand side. It's now routine to check that any normal form of t is a normal form of t' and vice versa.

□

Theorem 5.5 *The relation* $\Rightarrow_p$ *is confluent and given a term* t*, the set* $\mathtt{NF}_p(t)$ *consists of* $\Rightarrow_p$*-quasi normal forms*

Proof Given two reducts of t, a completion may be constructed to any element of the non-empty set $\mathtt{NF}_p(t)$. Now consider any reduct t' of t. By Proposition 5.4, $\mathtt{NF}_p(t') = \mathtt{NF}_p(t)$ so t' reduces to every element of $\mathtt{NF}_p(t)$. □

5.2 Decidability of the Conversion relation

Any $\Rightarrow_c$-reduction can be expressed as a sequence of $\Rightarrow_p$-reductions and $\beta_{+,+}$-commuting conversions. Thus the construction of $\Rightarrow_c$-quasi normal forms can be viewed as synthesising the $\Rightarrow_p$-normal forms just defined with the normal forms of the strongly normalising and confluent relation generated by $\beta_{+,+}$. This synthesis is a three stage process. Firstly all maximal conversions are recursively normalised, and then all $\beta_{+,+}$-commuting conversions performed. This produces terms all of whose conversions contain no potentially free conversions, and so all reducts of such terms must be $\Rightarrow_p$-reducts. Thus the construction of normal forms may be finished off by using the normalisation procedure for $\Rightarrow_p$. The relations $\mathtt{NF}$ and $\mathtt{NF}^o$ are defined to be the identity on terms containing no conversions and

$$\mathtt{NF}^o(t) = \{\beta_{+,+}(t[\sigma \leftarrow \alpha_\sigma])|\sigma \in \mathtt{MC}(t) \text{ and } \alpha_\sigma \in \mathtt{NF}(t/\sigma)\} \quad \text{and} \quad \mathtt{NF}(t) = \{\mathtt{NF}_p(t')|t' \in \mathtt{NF}^o(t)\}$$

The following conditions, although not sufficient, indicate the structure of $\Rightarrow_c$-quasi normal forms and are important in reasoning about them. A term is *stable* iff $\forall\sigma \in \mathtt{C}\ (t).\mathtt{PFC}\ (t/\sigma) = \emptyset$ and *distributed* iff either $0 \in \mathtt{C}\ (t)$ or $\mathtt{MPFC}(t) = \emptyset$ and all subterms are also distributed.

Lemma 5.6 *All $\Rightarrow_p$-reducts of stable terms are stable. Thus $\mathtt{NF}^o$ terms are stable and $\mathtt{NF}$ terms are both stable and distributed.*
Proof If $t \Rightarrow_p t'$ and there is a conversion in t' containing a potentially free sub-conversion, then there is also a conversion in t' containing a maximal potentially free sub-conversion. Thus by Proposition 5.4 there is a conversion in t also containing a maximal potentially free conversion and so the redex can't be stable.

For the second part, $\mathtt{NF}^o$ terms are shown stable by induction, while stability of $\mathtt{NF}$ terms now follows from the first part of the lemma. Finally $\mathtt{NF}$ terms are $\Rightarrow_p$-normal forms and so automatically distributed. □

Lemma 5.7 *If $t \Rightarrow_c t'$, then $\mathtt{NF}^o(t)$ and $\mathtt{NF}^o(t')$ are $\Rightarrow_p$-equivalent while the sets $\mathtt{NF}(t)$ and $\mathtt{NF}(t')$ are equal.*
Proof The second half of the lemma follows from Prop 5.4. For terms $\mathbf{case}(t, x.u, y.u)$ and u

$$\mathtt{NF}^o\mathbf{case}(t, x.u, y.u) = \beta_{+,+}\mathbf{case}(\mathtt{NF}t, x.\mathtt{NF}^o u, y.\mathtt{NF}^o u) =_p \mathtt{NF}^o u$$

If the reduction in question is a $\beta_{+,+}$-commuting conversion, the lemma is proved by direct calculation, while a basic expansion of maximal conversion $X \subseteq \mathtt{FC}(t)$ requires induction if $\mathtt{NF}(t/X)$ contains free conversions. Finally if the reduction is a congruence then the lemma follows by induction. □

Theorem 5.8 *The conversion relation is confluent, decidable and the set $\mathtt{NF}(t)$ consists of $\Rightarrow_c$-quasi normal forms of t.*
Proof The proof is as for the relation $\Rightarrow_p$. □

6 A Normalising Fragment

The second half of the decomposition of the expansionary rewrite relation is generated by β-redexes, commuting conversions and η-expansions. Since the normalisation proof shares many similarities with that in [13], we concentrate on the technical innovations required by the presence of coproducts. Because of the presence of commuting conversions, the proof is based on Prawitz notion of validity [16] as opposed to reducibility [7] as used before. The second innovation in the proof is required to cope with the failure of substitutivity. An operator $\mathcal{I}$, short for the identity, is introduced $\mathcal{I}t = (\lambda x.x)t$ and used to alter the traditional β-redexes as follows

Definition 6.1 *The reduction relation* $\Rightarrow_{\beta'}$ *is generated by the following basic redexes*

$$
\begin{array}{lrcl}
(\beta_{\times,i}) & \pi_0\langle u_0, u_1\rangle & \Rightarrow & u_1 \\
(\beta_{+,\times}) & \pi_i\mathbf{case}(t, x.u, y.v) & \Rightarrow & \mathbf{case}(t, x.\pi_i u, y.\pi_i v) \\
(\beta_{\rightarrow}) & (\lambda x.t)t' & \Rightarrow & t[t'/x] \\
(\beta_{+,\rightarrow}) & \mathbf{case}(t, x.u, y.v)t' & \Rightarrow & \mathbf{case}(t, x.ut', y.vt') \\
(\beta'_{+,1}) & \mathbf{case}(\mathbf{in}_1(t), x.u, y.v) & \Rightarrow & u[\mathcal{I}t/x] \\
(\beta'_{+,2}) & \mathbf{case}(\mathbf{in}_2(t), x.u, y.v) & \Rightarrow & v[\mathcal{I}t/y]
\end{array}
$$

and

$$
\begin{array}{ll}
(\beta_{+,+}) & \mathbf{case}(\mathbf{case}(t, x.u, y.v), x'.u', y'.v') \;\; \Rightarrow \\
\multicolumn{2}{l}{\mathbf{case}(t, x.\mathbf{case}(u, x'.u', y'.v'), y.\mathbf{case}(v, x'.u', y'.v'))}
\end{array}
$$

The rewrite relation obtained by replacing the altered β'*-redexes with the more traditional* β*-redexes of the introduction is denoted* $\Rightarrow_\beta$

Strong normalisation of $\Rightarrow_{\beta'}$ implies strong normalisation of $\Rightarrow_\beta$ and both relations share the same (unique) normal forms. To this relation is added limited possibilities for η-expansion. A term may be expanded providing it is neither an introduction term, nor a *case*-expressions of sum type and does not occur as the major premise of an elimination rule. The reduct of such an expansion is given by

$$
\eta(t) = \begin{cases}
t & \text{if } t \text{ is of base type} \\
* & \text{if } t \text{ is of unit type} \\
\lambda x.tx & \text{if } t \text{ is of function type} \\
\langle \pi_0 t, \pi_1 t\rangle & \text{if } t \text{ is of product type} \\
\mathbf{case}(t, x.\mathbf{in}_1(x), y.\mathbf{in}_2(y)) & \text{if } t \text{ is of sum type}
\end{cases}
$$

The context-sensitive nature of the restrictions on expansion manifest themselves in the definition of the relation simultaneously with a subrelation

Definition 6.2 *The relations* $\Rightarrow_{\mathcal{F}}$ *and* $\Rightarrow_{\mathcal{I}}$ *are defined simultaneously by the following inference rules*

$$
\frac{t \;\; \textit{expandable, not atomic type}}{t \Rightarrow_{\mathcal{F}} \eta(t)}
\qquad
\frac{t \Rightarrow_{\beta'} t'}{t \Rightarrow_{\mathcal{I}\cap\mathcal{F}} t'}
$$

$$
\frac{u \Rightarrow_{\mathcal{I}\cup\mathcal{F}} u'}{\langle u, v\rangle \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \langle u', v\rangle}
\qquad
\frac{v \Rightarrow_{\mathcal{I}\cup\mathcal{F}} v'}{\langle u, v\rangle \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \langle u, v'\rangle}
$$

$$
\frac{t \Rightarrow_{\mathcal{I}} t'}{\pi_0 t \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \pi_0 t'}
\qquad
\frac{t \Rightarrow_{\mathcal{I}} t'}{\pi_1 t \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \pi_1 t'}
$$

$$
\frac{t \Rightarrow_{\mathcal{I}} t'}{tu \Rightarrow_{\mathcal{I}\cap\mathcal{F}} t'u}
\qquad
\frac{u \Rightarrow_{\mathcal{I}\cup\mathcal{F}} u'}{tu \Rightarrow_{\mathcal{I}\cap\mathcal{F}} tu'}
$$

$$
\frac{t \Rightarrow_{\mathcal{I}\cup\mathcal{F}} t'}{\lambda x.t \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \lambda x.t'}
\qquad
\frac{t \Rightarrow_{\mathcal{I}} t'}{\mathbf{case}(t, x.u, y.v) \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \mathbf{case}(t', x.u, y.v)}
$$

$$
\frac{u \Rightarrow_{\mathcal{I}\cup\mathcal{F}} u'}{\mathbf{case}(t, x.u, y.v) \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \mathbf{case}(t, x.u', y.v)}
\qquad
\frac{v \Rightarrow_{\mathcal{I}\cup\mathcal{F}} v'}{\mathbf{case}(t, x.u, y.v) \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \mathbf{case}(t, x.u, y.v')}
$$

$$
\frac{t \Rightarrow_{\mathcal{I}\cup\mathcal{F}} t'}{\mathbf{in}_1(t) \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \mathbf{in}_1(t')}
\qquad
\frac{t \Rightarrow_{\mathcal{I}\cup\mathcal{F}} t'}{\mathbf{in}_2(t) \Rightarrow_{\mathcal{I}\cap\mathcal{F}} \mathbf{in}_2(t')}
$$

where $\mathcal{R} \in \{\mathcal{I}, \mathcal{F}\}$ *and* $\Rightarrow_{\mathcal{I}\cup\mathcal{F}}$, $\Rightarrow_{\mathcal{I}\cap\mathcal{F}}$ *are the union and intersection of* $\Rightarrow_{\mathcal{I}}$ *and* $\Rightarrow_{\mathcal{F}}$ *respectively.*

The relation $\Rightarrow_{\mathcal{I}}$ is the subrelation of $\Rightarrow_{\mathcal{F}}$ obtained by deleting top level expansions and so may be safely applied in situations where $\Rightarrow_{\mathcal{F}}$ may threaten to violate the context sensitive restrictions. The relations $\Rightarrow_{\mathcal{I}}$ and $\Rightarrow_{\mathcal{F}}$ are not congruences and this complicates traditional normalisation and confluence proofs. The major difficulty is the failure of substitutivity in that if $t \Rightarrow_{\mathcal{F}} t'$, there is in general no rewrite $t[u/x] \Rightarrow_{\mathcal{F}} t'[u/x]$, e.g. if u is an introduction term and the rewrite is induced by an expansion of a free occurrence of x in t. However the altered $\Rightarrow_{\beta'}$-redex substitutes an application term and so is immune from substitutivity problems. Of course this cannot be extended to function types as infinite reduction sequences would be formed $(\lambda x.x)(t) \Rightarrow x[\mathcal{I}t/x] = (\lambda x.x)(t)$, but fortunately the induction hypothesis in the strong normalisation proof is sufficient to cope with this latter case and so there is no need to alter $\beta_{\rightarrow}$. The relation $\Rightarrow_{\mathcal{F}}$ (although not $\Rightarrow_{\mathcal{I}}$) is locally confluent. This is a straight-forward generalisation of the equivalent proof in the calculus without coproducts and so is omitted

6.1 Strong Normalisation of $\Rightarrow_{\mathcal{F}}$

The relation $\Rightarrow_{\mathcal{F}}$ is proved strongly normalising by adapting the notion of validity [16] to cope with the presence of expansionary η-rewrites. The set of *valid* terms of type X, denoted $V(X)$ is defined to be the least non-empty type-indexed set of terms such that $V(X)/\Rightarrow_{\mathcal{I}} \subseteq V(X)$ and

- $\rightarrow R$: $\lambda x.t \in V(X \rightarrow Y)$ if $u \in V(X)$ then $t[u/x] \in V(Y)$
- $\times R$: $\langle u, v\rangle \in V(X \times Y)$ if $u \in V(X)$ and $v \in V(Y)$
- $+R$: $\mathbf{in}_1(t) \in V(A+B)$ if $t \in V(A)$
- $+R$: $\mathbf{in}_2(t) \in V(A+B)$ if $t \in V(B)$
- $+L$: $\mathbf{case}(t, x.u, y.v) \in V(X)$ if $u, v \in V(X)$
- $+L$: $\mathbf{case}(t, x.u, y.v) \in V(X_0 \times X_1)$ if $\mathbf{case}(t, x.\pi_i u, y.\pi_i v) \in V(X_i)$
- $+L$: $\mathbf{case}(t, x.u, y.v) \in V(X \rightarrow Y)$ if for any $w \in V(X)$ not containing x, y as free variables then $\mathbf{case}(t, x.uw, y.vw) \in V(Y)$

Notice that as variables have no $\Rightarrow_{\mathcal{I}}$-reducts, they are automatically valid. The last two clauses are required to deal with commuting conversions, but play a minor role in the proof. Before showing that valid terms satisfy strong closure properties, the reducts of a variable must be characterised.

Definition 6.3 *Define a function, Δ, which maps variables to sets of terms by induction over type structure*

$\Delta(z) = \{z\}$	*z is of base type*
$\Delta(z) = \{z, *\}$	*z is of unit type*
$\Delta(z) = \{z\} \cup \{\lambda x.t_1[zt_2/y] \mid t_1 \in \Delta(y), t_2 \in \Delta(x)\}$	*z is of exponent type*
$\Delta(z) = \{z\} \cup \{\langle t_0[\pi_0 z/x], t_1[\pi_1 z/y]\rangle \mid t_0 \in \Delta(x), t_1 \in \Delta(y)\}$	*z is of product type*
$\Delta(z) = \{z\} \cup \{\mathbf{case}(z, x.\mathbf{in}_1(t_1), y.\mathbf{in}_2(t_2)) \mid t_1 \in \Delta(x), t_2 \in \Delta(y)\}$	*z is of sum type*

where the variables x and y are of appropriate type. The function Δ is extended to terms by

$$\Delta(t) = \{t_0[t/z] \mid t_0 \in \Delta(z)\}$$

Lemma 6.4 *The function Δ gives the rewrites of a variable, i.e. $\Delta(z) = z/\Rightarrow^*_{\mathcal{F}}$*
Proof The proof is by induction on the type of the variable. That $\Delta(z) \subseteq z/\Rightarrow^*_{\mathcal{F}}$ follows by induction while, as $z \in \Delta(z)$, the reverse containment follows from showing $\Delta(z)/\Rightarrow_{\mathcal{F}} \subseteq \Delta(z)$. □

As usual in such normalisation proofs, valid terms are shown to satisfy certain closure properties

Definition 6.5 *Let P be a set of terms. Define the four predicates*

V1: If $t \in P$ then t is $\Rightarrow_{\mathcal{F}}$-strongly normalising
V2: If $t \in P$ then $t/\Rightarrow_{\mathcal{F}} \subseteq P$
V3: If $t \in P$ then $\Delta(t) \subseteq P$
V4: If $t \in P$ then $\mathcal{I}t \in P$

Although V4 has been included as a separate predicate, if P is the set of valid terms of some type, it is actually a consequence of the first three and so valid terms are shown, by induction over types, to satisfy only the first three predicates. Terms of base type cannot be expanded and so for such terms the relations $\Rightarrow_{\mathcal{I}}$ and $\Rightarrow_{\mathcal{F}}$ coincide. Thus valid terms of base type trivially satisfy the validity predicates. The only expansion of terms of unit type has as reduct the valid constant $*$ and so the validity predicates hold. The mathematics required for product and function types is essentially the same as in [13] and so omitted.

Quasi-introduction terms are generated by the syntax

$$q := \mathbf{in}_1(t) \mid \mathbf{in}_2(t) \mid \mathbf{case}(t, x.q, y.q)$$

and are closed under $\Rightarrow_{\mathcal{F}}$, are non-expandable and also the reduct of an η-expansion is a quasi-introduction term. They thus form a stepping stone in proving the validity predicates for terms of sum type. Let Arm be the function which maps a quasi-introduction term to the set of injections at its leafs.

Lemma 6.6 *If $V(X)$ and $V(Y)$ satisfy V1, V2 and V3, then the set of valid quasi-introduction terms of type $X+Y$ satisfies V1 and V2. Also if $u : X$ and $v : Y$ are valid, then so are $\mathbf{in}_1(u)$ and $\mathbf{in}_2(v)$.*

Proof Since quasi-introduction terms are non-expandable and closed under reduction, the first part of the lemma follows by induction on validity. The second half follows by induction on $|u|_{\mathcal{F}}$ and $|v|_{\mathcal{F}}$ □

Lemma 6.7 *Let $V(X)$ and $V(Y)$ satisfy V1, V2 and V3 and t be a valid term of type $X+Y$.*

- *If $t = \mathbf{case}(t_0, x.u, y.v)$, then $\mathbf{case}(t_0, x.\alpha, y.\beta)$ is valid, where $\alpha \in \eta(u)/\Rightarrow^*_{\mathcal{F}}$ and $\beta \in \eta(v)/\Rightarrow^*_{\mathcal{F}}$*
- *All terms $t' \in \Delta(t)$ are valid.*

Proof The proof is by simultaneous induction on the validity of t.

(i) By induction $\eta(u)$ and $\eta(v)$ are valid, quasi-introduction terms and hence so are α and β. Thus $|\alpha|_{\mathcal{F}} + |\beta|_{\mathcal{F}}$ is used as an inner induction rank. We are left to prove that all $\Rightarrow_{\mathcal{I}}$-reducts of $\mathbf{case}(t_0, x.\alpha, y.\beta)$ are valid. Those induced by reductions of proper subterms are valid by induction. This leaves two cases. If t_0 is an introduction term, and say $\mathbf{case}(\mathbf{in}_1(s), x.\alpha, y.\beta) \Rightarrow_{\mathcal{I}} \alpha[\mathcal{I}s/x]$ then by induction $\eta(u[\mathcal{I}s/x]) = \eta(u)[\mathcal{I}s/x]$ is valid quasi-introduction term. As $\alpha[\mathcal{I}s/x]$ is a reduct of this term it is also valid. Similarly if t_0 is a *case*-expression, the result of a commuting conversion is shown valid by applying induction to the reduct of the original term obtained by a commuting conversion.

(ii) We must prove that $\mathbf{case}(t, x.\mathbf{in}_1(u), y.\mathbf{in}_2(v))$ is valid where $u \in \Delta(x)$ and $v \in \Delta(y)$. By induction u and v are valid and strongly normalising and hence so are $\mathbf{in}_1(u)$ and $\mathbf{in}_2(v)$. These normalisation ranks then form an inner induction. Those 1-step $\Rightarrow_{\mathcal{I}}$ reducts induced by reductions of proper subterms are valid by induction, while the result of a top-level commuting conversion is valid by the first half of this lemma. Finally a reduction

$$\mathbf{case}(\mathbf{in}_1(t_0), x.\mathbf{in}_1(u), y.\mathbf{in}_2(v)) \Rightarrow_{\mathcal{F}} \mathbf{in}_1(u)[\mathcal{I}t_0/x]$$

is valid as by assumption t_0 is valid and of smaller type, so by induction so is $\mathcal{I}t_0$. Finally, by V3, so is $u[\mathcal{I}t_0/x]$ and hence $\mathbf{in}_1(u)[\mathcal{I}t_0/x]$

□

Corollary 6.8 *Let $V(X)$ and $V(Y)$ satisfy V1, V2 and V3. Then they hold for the set of terms $V(X+Y)$.*

Proof Let t be a term. The lemma is established by induction on the validity of t. All $\Rightarrow_{\mathcal{I}}$-reducts are valid and, by induction, strongly normalising. The only other reduct is a valid quasi-introduction term which is also strongly normalising. Thus all reducts of t are strongly normalising and hence so is t. The $\Rightarrow_{\mathcal{I}}$-reducts of a term are valid by definition, while the result of a basic expansion has already been shown valid. Finally V3 has just been established above. □

Having shown valid terms satisfy the validity predicates and are thus strongly normalising the last stage is to prove that all terms are valid. However before this can be done the criteria under which a **case**-expression may be shown to be valid must be simplified.

Lemma 6.9 *The term* $\mathbf{case}(t, x.u, y.v)$ *is valid iff* t *is strongly normalising,* u, v *are valid, and in addition if* $t' \in t/\Rightarrow_{\mathcal{I}}^{*}$ *is a quasi-introduction term and* $\mathbf{in}_1(\alpha) \in \mathrm{Arm}(t')$ *then* $u[\mathcal{I}\alpha/x]$ *is valid, and similarly for right injections.*

Proof The proof is by induction on firstly the type of the *case*-expression and secondly on $|t|_{\mathcal{F}} + |u|_{\mathcal{F}} + |v|_{\mathcal{F}}$. There are two proof obligations. Firstly if the *case*-expression is of sum or function type, the clauses pertaining to commuting conversions are easily established by induction. Secondly, those $\Rightarrow_{\mathcal{I}}$-reducts induced by reductions of subterms are valid by induction with the second part of the induction hypothesis following from substitutivity considerations, while a basic β-reduction has a valid reduct by assumption. Finally if t is a *case* expression then the result of a basic commuting conversion is shown valid by first using induction to prove the arms valid and then once more for the whole term. □

Finally all terms are shown valid in the traditional manner

Lemma 6.10 *Let* t *be a term and* θ *be a valid substitution. Then* $t\theta$ *is valid*

Proof The proof is by induction on t and follows the standard pattern. The only interesting part is for the term $\mathbf{case}(t, x.u, y.v)$. The terms $u\theta$, $v\theta$ and $t\theta$ are valid and thus strongly normalising by induction. Thus so is any $t' \in (t\theta)/\Rightarrow_{\mathcal{I}}^{*}$ and hence any $\mathbf{in}_1(\alpha) \in Arm(t')$. From this we may deduce α is also valid and so $\theta; x \mapsto \mathcal{I}\alpha$ is a valid substitution. Thus

$$(u\theta)[\mathcal{I}\alpha/x] = u(\theta; x \mapsto \mathcal{I}\alpha)$$

is a valid term □

Theorem 6.11 *The relations* $\Rightarrow_{\mathcal{F}}$ *and* $\Rightarrow_{\mathcal{I}}$ *are strongly normalising.*

Proof All terms are valid since the identity substitution is a valid substitution, and all valid terms are $\Rightarrow_{\mathcal{F}}$-strongly normalising. □

Lemma 6.12 *In a* $\Rightarrow_{\mathcal{F}}$*-normal form all subterms are either of base type, (quasi-)introduction terms or occur as the major premise of an elimination rule.* λ*-abstractions or are applied.*

Proof Simple induction over term structure □

Finally the whole of the expansionary rewrite relation embeds into the conversion relation of the previous section.

Theorem 6.13 *If* t *and* t' *are* $\Rightarrow_{c}$*-equivalent, then so are their* $\Rightarrow_{\mathcal{F}}$*-normal forms. Hence the expansionary rewrite relation is both confluent and decidable.*

Proof The lemma is proved by calculation the effect on conversions of $\Rightarrow_{\mathcal{F}}$-rewrites on t and t'.

Decidability and confluence of the expansionary rewrite relation may now be lifted from the same results for the conversion relation. □

7 Conclusions and Further Work

In this paper an extensional equality for terms of BCC^{-} was given. To each term we can associate a finite set of normal forms, calculable in two stages; firstly by $\beta\eta$-normalisation and secondly by expanding as many conversions as possible. As terms equivalent in the equational theory have the same set of normal forms, comparison of these normal forms provides a decision procedure for equality of terms.

There are two principle directions in which this research may be extended. Firstly the inability to define a unique normal form is closely linked to form of the *case*-expression which permits the elimination of one term at a time. An alternate, parallel elimination, allowing the concurrent elimination of several terms should permit the definition of unique normal forms and this is the subject of current work.

In a different direction, extensionality principles can be applied to other theories, e.g. those with recursive and dependent types. However the nature of the resulting proof theory is still vague and requires considerable research.

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